

# A strong form of Arnold diffusion for two and a half degrees of freedom

V. Kaloshin\*, K. Zhang†

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## Abstract

In the present paper we prove a strong form of Arnold diffusion. Let  $\mathbb{T}^2$  be the two torus and  $B^2$  be the unit ball around the origin in  $\mathbb{R}^2$ . Fix  $\rho > 0$ . Our main result says that for a “generic” time-periodic perturbation of an integrable system of two degrees of freedom

$$H_0(p) + \varepsilon H_1(\theta, p, t), \quad \theta \in \mathbb{T}^2, \quad p \in B^2, \quad t \in \mathbb{T} = \mathbb{R}/\mathbb{Z},$$

with a strictly convex  $H_0$ , there exists a  $\rho$ -dense orbit  $(\theta_\varepsilon, p_\varepsilon, t)(t)$  in  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$ , namely, a  $\rho$ -neighborhood of the orbit contains  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$ .

Our proof is a combination of geometric and variational methods. The fundamental elements of the construction are usage of crumpled normally hyperbolic invariant cylinders from [13], flower and simple normally hyperbolic invariant manifolds from [47] as well as their kissing property at a strong double resonance. This allows us to build a “connected” net of 3-dimensional normally hyperbolic invariant manifolds. To construct diffusing orbits along this net we employ a version of Mather variational method [54] equipped with weak KAM theory [34], proposed by Bernard in [9].

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\*University of Maryland at College Park (vadim.kaloshin@gmail.com)

†University of Toronto (kzhang@math.utoronto.edu)

# Contents

<b>1</b>	<b>Introduction</b>	<b>5</b>
1.1	Statement of the result . . . . .	5
<b>2</b>	<b>Preliminary splitting of the proof of Theorem 1 into ten Key Theorems</b>	<b>11</b>
2.1	Decomposition of the diffusion path into single and double resonances.	11
2.2	Genericity conditions at single resonances . . . . .	13
<b>3</b>	<b>Theorems on existence of normally hyperbolic invariant manifolds</b>	<b>19</b>
3.1	Description of single resonances . . . . .	19
3.2	Description of double resonances and generic properties at high energy	22
3.3	Description of double resonances (low energy) . . . . .	25
3.4	Generic properties of homoclinic orbits and genericity at low energy .	27
3.5	Heuristics of the diffusion across double resonances . . . . .	34
3.5.1	Crossing through along a simple loop $\gamma_h^0 \ni 0$ . . . . .	34
3.5.2	Crossing through along a simple loop $\gamma_h^0 \not\ni 0$ . . . . .	35
3.5.3	Crossing through along a non-simple loop $\gamma_h^0$ . . . . .	35
3.5.4	Turning a corner from $\Gamma$ to $\Gamma'$ . . . . .	35
<b>4</b>	<b>Localization of the Aubry sets and the Mañe sets and Mather's projected graph theorems</b>	<b>37</b>
4.1	Localization and Mather's projected graph theorem for single resonances	37
4.2	Localization and Mather's projected graph theorem for double resonances	40
4.2.1	NHIMs near a double resonance . . . . .	41
4.2.2	Choice of the cohomology classes . . . . .	42
4.2.3	Properties of the Aubry sets and Mañe sets . . . . .	44
4.3	Choice of auxiliary cohomology classes for a non-simple homology . .	46
<b>5</b>	<b>Description of <math>c</math>-equivalence and a variational <math>\lambda</math>-lemma</b>	<b>49</b>
5.1	Heuristic descriptions . . . . .	49
5.2	Forcing relation and shadowing . . . . .	50
5.3	Choice of cohomology classes for global diffusion . . . . .	52
<b>6</b>	<b>Equivalent forcing classes</b>	<b>56</b>
6.1	Equivalent forcing class along single resonances . . . . .	56
6.2	Equivalent forcing class along cylinders of the same homology class .	58
6.3	Equivalent forcing class between kissing cylinders . . . . .	63

<b>7</b>	<b>Normally hyperbolic invariant cylinders through the transition zone into double resonances</b>	<b>65</b>
7.1	Normally hyperbolic invariant manifolds going into double resonances	66
7.2	A Normal form in the transition zone . . . . .	68
7.2.1	Autonomous case and slow-fast coordinates . . . . .	68
7.2.2	Time-periodic setting and reduction to $d = 3$ . . . . .	72
7.3	Construction of an isolating block . . . . .	73
7.3.1	Auxiliary estimates on the vector field. . . . .	73
7.3.2	Constructing the isolation block . . . . .	78
<b>8</b>	<b>Proof of Key Theorem 3 about existence of invariant cylinders at double resonances</b>	<b>81</b>
8.1	Normal form near the hyperbolic fixed point . . . . .	82
8.2	Behavior of a family of orbits passing near 0 and Shil'nikov boundary value problem . . . . .	84
8.3	Properties of the local maps . . . . .	88
8.4	Conley-McGehee isolating blocks . . . . .	93
8.5	Single leaf cylinders . . . . .	95
8.6	Double leaf cylinders . . . . .	99
8.7	Normally hyperbolic invariant cylinders the slow mechanical system .	100
8.8	Smooth approximations . . . . .	103
8.9	Proof of Lemma 3.3 on cyclic concatenations of simple geodesics . . .	104
<b>9</b>	<b>Diffusion mechanisms, weak KAM theory and forcing relation</b>	<b>106</b>
9.1	Duality of Hamiltonian and Lagrangian, homology and cohomology .	106
9.2	Overlapping pseudographs . . . . .	108
9.3	Evolution of pseudographs and the Lax-Oleinik mapping . . . . .	109
9.4	Aubry, Mather, Mañe sets in Hamiltonian setting and properties of forcing relation . . . . .	110
9.5	Symplectic invariance of the Mather, Aubry, and Mañe sets . . . . .	112
9.6	Mather's $\alpha$ and $\beta$ -functions, Legendre-Fenchel transform and barrier functions . . . . .	112
9.7	Forcing relation of cohomology classes and its dynamical properties .	114
9.8	Other diffusion mechanisms and apriori unstable systems . . . . .	115
<b>10</b>	<b>Properties of the barrier functions</b>	<b>117</b>
10.1	The Lagrangian setting . . . . .	117
10.2	Properties of the action and barrier functions . . . . .	122
10.2.1	Uniform family and Tonelli convergence . . . . .	122
10.2.2	Properties of the action function . . . . .	123

10.2.3	Properties of the barrier function . . . . .	125
10.2.4	Semi-continuity of the Aubry and Mañe set and continuity of the barrier function . . . . .	126
<b>11</b>	<b>Diffusion along the same homology at double resonance</b>	<b>129</b>
11.1	Localization and graph theorem . . . . .	129
11.2	Forcing relation along the same homology class . . . . .	131
11.3	Local extensions of the Aubry sets and a proof of Theorem 23 . . . .	134
11.4	Generic property of the Aubry sets $\tilde{\mathcal{A}}_{H_\varepsilon}(\bar{c}_h(E))$ . . . . .	137
11.5	Generic property of the $\alpha$ -function and proof of Theorem 13 . . . . .	137
11.6	Nondegeneracy of the barrier functions . . . . .	138
<b>12</b>	<b>Equivalent forcing class between kissing cylinders</b>	<b>143</b>
12.1	Variational problem for the slow mechanical system . . . . .	143
12.2	Variational problem in the original coordinates . . . . .	146
12.3	Scaling limit of the barrier function . . . . .	148
12.4	Proof of forcing relation . . . . .	150
<b>A</b>	<b>Generic properties of mechanical systems on the two-torus</b>	<b>155</b>
A.1	Generic properties of periodic orbits . . . . .	155
A.2	Generic properties of minimal orbits . . . . .	161
A.3	Proof of Theorem 4 about genericity of [DR1]-[DR3] . . . . .	168
A.4	Proof of Theorem 5 . . . . .	170
<b>B</b>	<b>Derivation of the slow mechanical system</b>	<b>172</b>
B.1	Normal forms near double resonances . . . . .	172
B.2	Affine coordinate change and rescaling . . . . .	177
B.3	Variational properties of the coordinate changes . . . . .	180
<b>C</b>	<b>Variational aspects of the slow mechanical system</b>	<b>184</b>
C.1	Relation between the minimal geodesics and the Aubry sets . . . . .	184
C.2	Characterization of the channel and the Aubry sets . . . . .	187
C.3	The width of the channel . . . . .	189
C.4	The case $E = 0$ . . . . .	191
<b>D</b>	<b>Transition between single and double resonance</b>	<b>195</b>
<b>E</b>	<b>Notations</b>	<b>197</b>

# 1 Introduction

The famous question called the ergodic hypothesis, formulated by Maxwell and Boltzmann, suggests that for a typical Hamiltonian on a typical energy surface all, but a set of zero measure of initial conditions, have trajectories dense in this energy surface. However, KAM theory showed that for an open set of nearly integrable systems there is a set of initial conditions of positive measure with almost periodic trajectories. This disproved the ergodic hypothesis and forced to reconsider the problem.

A quasi-ergodic hypothesis, proposed by Ehrenfest [33] and Birkhoff [17], asks if a typical Hamiltonian on a typical energy surface has a dense orbit. A definite answer whether this statement is true or not is still far out of reach of modern dynamics. There was an attempt to prove this statement by E. Fermi [36], which failed (see [37] for a more detailed account). To simplify the problem, Arnold [5] asks:

*Does there exist a real instability in many-dimensional problems of perturbation theory when the invariant tori do not divide the phase space?*

For nearly integrable systems of one and a half and two degrees of freedom the invariant tori do divide the phase space and an energy surface respectively. This implies that instability do not occur. We solve a weaker version of this question for systems with two and a half and 3 degrees of freedom. This corresponds to time-periodic perturbations of integrable systems with two degrees of freedom and autonomous perturbations of integrable systems with three degrees of freedom.

## 1.1 Statement of the result

Let  $(\theta, p) \in \mathbb{T}^2 \times B^2$  be the phase space of an integrable Hamiltonian system  $H_0(p)$  with  $\mathbb{T}^2$  being 2-dimensional torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \ni \theta = (\theta_1, \theta_2)$  and  $B^2$  being the unit ball around 0 in  $\mathbb{R}^2$ ,  $p = (p_1, p_2) \in B^2$ . Assume that  $H_0$  is strictly convex, i.e. Hessian  $\partial_{p_i p_j}^2 H_0$  is strictly positive definite.

Consider a smooth time periodic perturbation

$$H_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon H_1(\theta, p, t), \quad t \in \mathbb{T} = \mathbb{R}/\mathbb{T}.$$

We study a strong form of Arnold diffusion for this system, namely,

*existence of orbits  $\{(\theta_\varepsilon, p_\varepsilon)(t)\}_t$  going from one open set  $p_\varepsilon(0) \in U$  to another  $p_\varepsilon(t) \in U'$  for some  $t = t_\varepsilon > 0$ .*

Arnold [3] proved existence of such orbits for an example and conjectured that they exist for a typical perturbation (see e.g. [4, 5, 7]).

Integer relations  $\vec{k} \cdot (\partial_p H_0, 1) = 0$  with  $\vec{k} = (\vec{k}_1, k_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$  and  $\cdot$  being the inner product define a *resonant segment*. The condition that the Hessian of  $H_0$  is

non-degenerate implies that  $\partial_p H_0 : B^2 \rightarrow \mathbb{R}^2$  is a diffeomorphism and each resonant line defines a smooth curve embedded into action space

$$\Gamma_{\vec{k}} = \{p \in B^2 : \vec{k} \cdot (\partial_p H_0, 1) = 0\}.$$

If curves  $\Gamma_{\vec{k}}$  and  $\Gamma_{\vec{k}'}$  are given by two linearly independent resonances vectors  $\{\vec{k}, \vec{k}'\}$ , they either have no intersection or intersect at a single point in  $B^2$ .

We call a vector  $\vec{k} = (k_1, k_0) = (k_1^1, k_1^2, k_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$  and the corresponding resonance  $\Gamma = \Gamma_{\vec{k}}$  *space irreducible* if either  $(k_1^1, k_1^2) = (1, 0)$  or  $(0, 1)$  or  $\gcd(k_1^1) = 1$ , i.e.  $|k_1^1|$  and  $|k_1^2|$  are relatively prime.

Consider now two open sets  $U, U' \subset B^2$ . Select a finite collection of space irreducible resonant segments  $\{\Gamma_j = \Gamma_{\vec{k}_j}\}_{j=1}^N$  for some collection of  $\{\vec{k}_j\}_{j=1}^N$

- with neighbors  $\vec{k}_j$  and  $\vec{k}_{j+1}$  being linearly independent,
- $\Gamma_j \cap \Gamma_{j+1} \neq \emptyset$  for  $j = 1, \dots, N-1$  and so that
- $\Gamma_1 \cap U \neq \emptyset$  and  $\Gamma_N \cap U' \neq \emptyset$ .

We would like to construct diffusing orbits along a connected path formed by segments inside  $\Gamma_j$ 's, i.e. we select a connected piecewise smooth curve  $\Gamma^* \subset \cup_{j=1}^N \Gamma_j$  so that  $\Gamma^* \cap U \neq \emptyset$  and  $\Gamma^* \cap U' \neq \emptyset$  (see Figure 2).

Consider the space of  $C^r$  perturbations  $C^r(\mathbb{T}^2 \times B^2 \times \mathbb{T})$  with a natural  $C^r$  norm given by maximum of all partial derivatives of order up to  $r$ , here  $r < +\infty$ . Denote by  $\mathcal{S}^r = \{H_1 \in C^r(\mathbb{T}^2 \times B^2 \times \mathbb{T}) : \|H_1\|_{C^r} = 1\}$  the unit sphere in this space.

**Theorem 1.** *In the above notations fix the piecewise smooth segment  $\Gamma^*$  and  $4 \leq r < +\infty$ . Then there is an open and dense set  $\mathcal{U} = \mathcal{U}_{\Gamma^*} \subset \mathcal{S}^r$  and a nonnegative function  $\varepsilon_0 = \varepsilon_0(H_1)$  with  $\varepsilon_0|_{\mathcal{U}} > 0$ . Let  $\mathcal{V} = \{\epsilon H_1 : H_1 \in \mathcal{U}, 0 < \epsilon < \varepsilon_0\}$ , then for an open and dense set of  $\epsilon H_1 \in \mathcal{W} \subsetneq \mathcal{V}$  the Hamiltonian system  $H_\epsilon = H_0 + \epsilon H_1$  has an orbit  $\{(\theta_\epsilon, p_\epsilon)(t)\}_t$  whose action component satisfies*

$$p_\epsilon(0) \in U, \quad p_\epsilon(t) \in U' \quad \text{for some } t = t_\epsilon > 0$$

Moreover, for all  $0 < t < t_\epsilon$  the action component  $p_\epsilon(t)$  stays  $O(\sqrt{\epsilon})$ -close to the union of resonances  $\Gamma^*$ .

**Remark 1.1.** *The open and dense set of perturbation  $\mathcal{U} \subset \mathcal{S}^r$  will be defined by three sets of non-degeneracy conditions.*

- In section 2.2, conditions [G0]-[G2] defines the quantitative non-degeneracy condition for  $\lambda > 0$ . The set of  $\lambda$ -non-degenerate perturbations  $\mathcal{U}_{SR}^\lambda$  is open, and the union  $\bigcup_{\lambda>0} \mathcal{U}_{SR}^\lambda$  is open and dense.

- Each  $\lambda > 0$  determines a finite set of double resonances on  $\Gamma^*$ . For each double resonance  $p_0$ , we define two sets of non-degeneracy conditions.

In section 3.2, conditions [DR1]-[DR3] defines the non-degeneracy at a double resonance  $p_0$  away from singularities, the non-degenerate set  $\mathcal{U}_{DR}^E(p_0)$  is open and dense.

In section 3.4, conditions [A0]-[A4] defines the non-degeneracy at double resonances  $p_0$  near singularities. The non-degenerate set  $\mathcal{U}_{DR}^{Crit}(p_0)$  is open and dense.

We have

$$\mathcal{U} = \bigcup_{\lambda > 0} \mathcal{U}_{SR}^\lambda \cap \left( \bigcap_{p_0} \mathcal{U}_{DR}^E(p_0) \cap \mathcal{U}_{DR}^{Crit}(p_0) \right),$$

where  $p_0$  ranges over all double resonances. This set is clearly open and dense.

For any  $H_1 \in \mathcal{U}$  we prove that the Hamiltonian  $H_0 + \epsilon H_1$  has a connected collection of 3-dimensional normally hyperbolic (weakly) invariant manifolds in the phase space  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$  which “shadow” a collection of segments  $\Gamma^*$  connecting  $U$  and  $U'$  in the sense that the natural projection of these manifolds onto  $B^2$  is  $O(\sqrt{\epsilon})$ -close to each point in  $\Gamma^*$ . Note that Marco [49, 50] announced a similar result.

Once this structure established we impose further non-degeneracy condition  $\mathcal{W} \subsetneq \mathcal{V}$ . For each  $\epsilon H_1 \in \mathcal{W}$  we prove existence of orbits diffusing along this collection of invariant manifolds. Our proof relies on Mather variational method [54] equipped with weak KAM theory of Fathi [34]. The crucial element is Bernard’s notion [9] of forcing relation (see section 5.2 for more details).

Notice that the notion of genericity we use is not standard. We show that in a neighborhood of perturbations of  $H_0$  the set of good directions  $\mathcal{U}$  is open dense in  $\mathcal{S}^r$ . Around each exceptional (nowhere dense) direction we remove a cusp and call the complement  $\mathcal{V}$ . For this set of perturbations we establish connected collection of invariant manifolds. Then in the complement to some exceptional perturbations  $\mathcal{W}$  we show that there are diffusing orbits “shadowing” these cylinders. Mather calls such a set of perturbations *cusp residual*. See Figure 1.

## 1. Autonomous version

Let  $n = 3$ ,  $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \in B^3$ , and  $\tilde{H}_0(p)$  be a strictly convex Hamiltonian. Consider the region  $\partial_{p_i} H_0 > \rho > 0$  for some  $\rho > 0$  and two open sets  $U$  and  $U'$  in this region. Then for a generic (autonomous) perturbation  $\tilde{H}_0(\tilde{p}) + \epsilon \tilde{H}_1(\tilde{\theta}, \tilde{p})$  there is an orbit  $(\tilde{\theta}_\epsilon, \tilde{p}_\epsilon)(t)$  whose action component connects  $U$  with  $U'$ , namely,

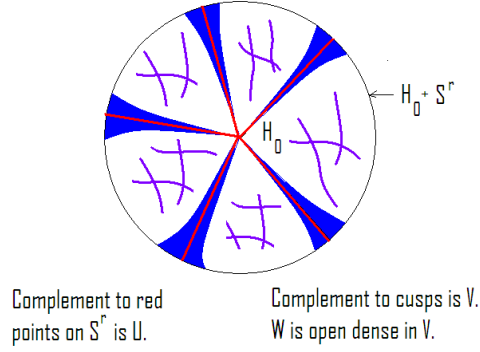


Figure 1: Description of generic perturbations

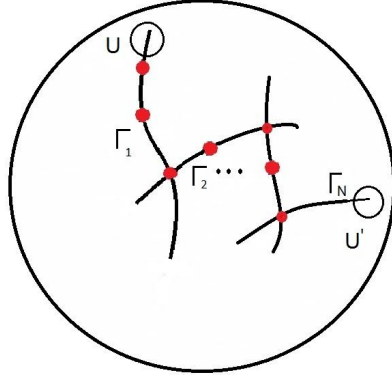


Figure 2: Resonant net

$\tilde{p}_\varepsilon(0) \in U$  and  $\tilde{p}_\varepsilon(t) \in U'$  for some  $t = t_\varepsilon$ . This can be proved using energy reduction to time periodic system of two and a half degrees of freedom (see e.g. [6, Section 45]).

## 2. Generic instability of resonant totally elliptic points

In [44] stability of resonant totally elliptic fixed points of symplectic maps in dimension 4 is studied. It is shown that generically a convex, resonant, totally elliptic point of a symplectic map is Lyapunov unstable.

## 3. Relation with Mather's approach

Theorem 1 was announced by Mather [56]. Some parts are the proof of written in [57]. Our approach is quite different from the one initiated in these two



manuscripts. Here we summarize the differences:

- We start by constructing a net of normally hyperbolic invariant cylinders “over” resonant segments  $\Gamma^*$ .
- We prove that certain variationally defined invariant (Aubry) sets belongs to these cylinders.
- We show that these sets have the same structure as Aubry-Mather sets for twist maps.
- In a single resonance, mainly done in [13], the fact that these invariant sets belong to invariant cylinders allows us to adapt techniques from [9, 23, 24]. Namely, we apply variational techniques for proving existence of Arnold diffusion for a priori unstable systems (see section 9.8 for more details).
- One important obstacle is the problem of regularity of barrier functions (see section 9), which outside of the realm of twist maps is difficult to overcome. It is our understanding that Mather [62] handles this problem without proving existence of invariant cylinders.
- In a double resonance we also construct normally hyperbolic invariant cylinders. This leads to a fairly simple and explicit structure of minimal orbits near a double resonance. In particular, in order to switch from one resonance to another we need *only one jump* (see section 3.5 for an heuristic explanation and Key Theorem 10 for the precise claim).
- It is our understanding that Mather’s approach [62] requires an implicitly defined number of jumps. His approach resembles his proof of existence of diffusing orbits for twist maps inside a Birkhoff region of instability [55].

#### 4. Convexity of $H_0$ on an open set

Suppose  $H_0(p)$  is strictly convex on an open set

$$Conv = \{p \in B^2 : \text{Hessian } \partial_{p_i p_j}^2 H_0(p) \text{ is strictly positive definite}\}.$$

Then for any connected component  $Conv' \subset Conv$  Theorem 1 applies. More exactly, for any pair of open sets  $U, U' \subset Conv'$  fix a smooth segment  $\Gamma^* \subset Conv'$  and  $r \geq 4$ . Then there is an open and dense set  $\mathcal{U} = \mathcal{U}_{\Gamma^*} \subset \mathcal{S}^r$  and a nonnegative function  $\varepsilon_0 = \varepsilon_0(H_1)$  with  $\varepsilon_0|_{\mathcal{U}} > 0$ . Let  $\mathcal{V} = \{\epsilon H_1 : H_1 \in \mathcal{U}, 0 < \epsilon < \varepsilon_0\}$ , then for an open and dense set of  $\epsilon H_1 \in \mathcal{V}$  the Hamiltonian system  $H_\epsilon = H_0 + \epsilon H_1$  has an orbit  $\{(\theta_\epsilon, p_\epsilon)(t)\}_t$  whose action component satisfies  $p_\epsilon(0) \in U$ ,  $p_\epsilon(t) \in U'$  for some  $t = t_\epsilon > 0$ .

Notice that Theorem 1 guarantees that this orbit  $\{(\theta_\epsilon, p_\epsilon)(t)\}_t$  has action component  $O(\sqrt{\epsilon})$  close to  $\Gamma^*$ . This is a convexity region for  $H_0$ . Therefore, one can extend  $H_0$  to another Hamiltonian  $\tilde{H}_0$ , which is convex outside of  $Conv$ . Then apply Theorem 1 and notice that under the condition that action component  $p \in Conv$  orbits of  $H_0 + \epsilon H_1$  and  $\tilde{H}_0 + \epsilon H_1$  coincide.

## 5. Non-convex Hamiltonians

In the Hamiltonian  $H_0$  is non-convex for all  $p \in B^2$ , for example,  $H_0(p) = p_1^2 - p_2^2$ , the problem of Arnold diffusion is wide open. With our approach it is closely related to another deep open problem of extending Mather theory and weak KAM theory beyond convex Hamiltonians.

## 6. The Main Result

One interesting application of Theorem 1 is the following Theorem, which is our main result.

**Theorem 2.** *For any  $\rho > 0$  and any  $r \geq 4$  there is an open and dense set  $\mathcal{U} = \mathcal{U}_\rho \subset \mathcal{S}^r$  and a nonnegative function  $\varepsilon_0 = \varepsilon_0(H_1, \rho)$  with  $\varepsilon_0|_{\mathcal{U}} > 0$ . Let  $\mathcal{V} = \{\epsilon H_1 : H_1 \in \mathcal{U}, 0 < \epsilon < \varepsilon_0\}$ , then for an open and dense set of  $\epsilon H_1 \in \mathcal{W} \subsetneq \mathcal{V}$  the Hamiltonian system  $H_\epsilon = H_0 + \epsilon H_1$  has a  $\rho$ -dense orbit  $\{(\theta_\epsilon, p_\epsilon, t)(t)\}_t$  in  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$ , namely, its  $\rho$ -neighborhood contains  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$ .*

**Remark 1.2.** *To prove this theorem one needs to take a finite set of resonant lines  $\{\Gamma_j = \Gamma_{\tilde{k}_j}\}_{j=1}^N$  so that  $\Gamma^* = \cup_{j=1}^N \Gamma_j$  is  $\rho/3$ -dense in the action domain  $B^2$ . Moreover, unperturbed resonant orbits for each  $p \in \Gamma^*$  have angular components  $\theta(t) = \theta(0) + tp \pmod{1}$  that are  $\rho/3$ -dense in  $\mathbb{T}^2$ . Since  $\dot{p} = O(\epsilon)$  and by Theorem 1 for diffusing orbits action  $p_\epsilon(t)$  is  $O(\sqrt{\epsilon})$ -close to  $\Gamma^*$ , we have  $\rho$ -density of  $(\theta_\epsilon, p_\epsilon, t)(t)$  in  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$ .*

*As the number of resonant lines  $N$  increases, the size of admissible perturbation  $\varepsilon_0(H_1, \rho)$  goes to zero. Examples of Hamiltonians having orbits accumulating to “large” sets and shadow infinitely many resonant lines are proposed in [45, 46].*

The proof of the main Theorem naturally divides into two major parts:

- **I Geometric:** construct a connected net of normally hyperbolic invariant manifolds (NHIMs) along the selected resonant lines  $\Gamma^*$ .
- **II Variational:** construct orbits diffusing along this net.

The idea to construct a connected net of NHIMs appeared in Kaloshin-Zhang-Zheng [46]. [46] describes the construction of an example of a  $C^\infty$ -Hamiltonian  $H$  in a small  $C^k$ -neighborhood  $k \geq 2$  of  $H_0(p) = \frac{1}{2}\|p\|^2$  such that, on  $H^{-1}(1/2)$ , the Hamiltonian flow of  $H$  has an orbit dense in a set of positive measure.

## 2 Preliminary splitting of the proof of Theorem 1 into ten Key Theorems

It turns out that for action  $p$  near the diffusion path  $\Gamma^*$  there are two types of qualitative behavior. We start presenting our strategy of the proof by dividing  $\Gamma^*$  into two parts: *single and (strong) double resonances*. Once we make this decomposition we impose three sets of non-degeneracy conditions a perturbation  $\varepsilon H_1$ .

The first set is given in section 2.2, conditions [G0]-[G2] and leads to generic existence of certain (crumpled) normally hyperbolic invariant cylinders near single resonances.

It turns out that near double resonances there are also two types of qualitative behavior: away from singularity (high energy) and near singularity (low energy). For high energy in section 3.2 we impose conditions [DR1]-[DR3] and also exhibit existence of certain normally hyperbolic invariant cylinders. Then for low energy in section 3.4 we define conditions [A0]-[A4] and describes more sophisticated normally hyperbolic invariant manifolds.

These three sets of conditions lead to a definition of the set  $\mathcal{U} \subset \mathcal{S}^r$  and reveals a connected net of normally hyperbolic invariant manifolds. Once existence of these invariant manifolds is established we prove existence of orbits diffusing along them under additional non-degeneracy conditions. More precisely, we divide the proof of the main result into 10 (Key) Theorems.

- The first three establish existence of normally hyperbolic invariant manifolds in each regime.

- The next four shows that carefully chosen family of invariant (Aubry) sets belongs to these manifolds and satisfy a certain (Mather's projected) graph property. To some extent, these invariant sets are like Aubry-Mather sets for twist maps.

- The last three shows that there are orbits diffusing along these family of Aubry sets, provided additional non-degeneracy hypothesis are satisfied.

Theorems involved into this scheme are called *Key Theorems*. The body of the proof contains other claims, including Lemmas, Propositions, and Theorems, used to prove Key Theorems.

### 2.1 Decomposition of the diffusion path into single and double resonances.

Diffusion in Theorem 1 takes place along  $\Gamma^*$ . The first important step is to decompose  $\Gamma^*$  into two sets. This decomposition corresponds to two type of qualitative dynamic behavior. To do this division we need to start with the unperturbed dynamics:

$$\dot{\theta} = \partial_p H_0(p), \quad \dot{p} = 0.$$

Fix a large integer  $K$  independent of  $\varepsilon$  and to be specified later. For each action  $p \in \Gamma^*$  it belongs to one of resonant segments, i.e.  $p \in \Gamma_j = \Gamma_{\vec{k}_j}$  for some  $j = 1, \dots, N$ . For  $p \in \Gamma_j$  define a *slow angle*  $\theta_j^s = \vec{k}_j \cdot (\theta, t)$ . We say that for  $p \in \Gamma^*$  we have

1. *K-single resonance* if there is exactly one slow angle  $\theta^s$  and all others are not:  $\vec{k} \cdot (\partial_p H_0(p), 1) = \vec{k} \cdot (\dot{\theta}, \dot{t}) \neq 0$  for each  $\vec{k} \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$  with  $|\vec{k}| \leq K$ .
2. *K-(strong) double resonance* if there is exactly two<sup>1</sup> slow angle  $\theta_1^s = \vec{k} \cdot (\theta, t)$  and  $\theta_2^s = \vec{k}' \cdot (\theta, t)$  with  $\vec{k}, \vec{k}' \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$  with  $|\vec{k}|, |\vec{k}'| \leq K$ .

We omit dependence on  $K$  for brevity.

Fix  $j \in \{1, \dots, N\}$ , the corresponding resonance vector  $\vec{k}_j \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$ , and the resonant segment  $\Gamma_j = \Gamma_{\vec{k}_j} \cap \Gamma^*$ . For other  $j$ 's the decomposition procedure is the same. Let  $\theta^s = \vec{k}_j \cdot (\theta, t)$  be the slow angle as above. In defining a fast angle there is some freedom. Let  $\vec{k}' = (\vec{k}'_j, k'_0) \nparallel \vec{k}_j$  and chosen so that

$$\det B = 1, \quad B = \begin{bmatrix} \vec{k}_j \\ \vec{k}'_j \end{bmatrix},$$

with  $\vec{k}_j$  being the  $\mathbb{Z}^2$ -component of  $\vec{k}_j$  and  $\vec{k}_j, \vec{k}'_j$  viewed as row vectors<sup>2</sup>.

The coordinate change can be completed to a symplectic one by considering the extended phase space  $(\theta, t, p, E)$ . Define  $p' = (p^s, p^f)$  and  $E'$  by the relation satisfying  $p = B^T p'$  and  $E = (k_0, k'_0) \cdot p' + E'$ , (see section B), then

$$L_j : (\theta, t, p, E) \mapsto (\theta^s, \theta^f, t, p', E')$$

is symplectic. By a direct calculation, we have the  $(\theta, t, p)$  components of  $L_j^{-1}$  is independent of  $E$ , so we can treat  $L_j^{-1}$  as a map from  $(\theta^s, \theta^f, t, p')$  to  $(\theta, t, p)$ .

Due to non-degeneracy of the Hessian of  $H_0$  we can use  $p^f$  as a smooth parametrization of  $\Gamma_j$ , i.e.

$$\Gamma_j = \{(p_*^s(p^f), p^f) : p^f \in [a_{min}^j, a_{max}^j]\},$$

where values  $a_{min}^j$  and  $a_{max}^j$  correspond to the end points of  $\Gamma_j$ . It is natural to define *the averaged potential*

$$Z_j(\theta^s, p) = \int \int H_1 \circ L_j^{-1}(\theta^s, p^s, \theta^f, p^f, t) d\theta^f dt. \quad (1)$$

When there is no confusion we omit dependence on  $j$  to keep notations simpler.

<sup>1</sup>Since  $\dot{t} = 1$  and  $k_0 \neq 0$  there can't be three slow angles.

<sup>2</sup>Choice of  $\vec{k}'_j$  is not unique

## 2.2 Genericity conditions at single resonances

Call a value  $p^f$  on  $\Gamma$  *regular* if  $Z(\theta^s, p_*^s(p^f))$  has a unique non-degenerate global maximum on  $\mathbb{T}^s \cong \mathbb{T} \ni \theta^s$  at some  $\theta_*^s = \theta^s(p^f)$ . We say the maximum is *non-degenerate* if the Hessian of  $Z$  with respect to  $\theta^s$  is strictly negative definite.

Call a value  $p^f$  on  $\Gamma_j$  *bifurcation* if  $Z(\theta^s, p_*^s(p^f))$  has exactly two global non-degenerate maxima on  $\mathbb{T}^s \ni \theta^s$  at some  $\theta_1^s = \theta_1^s(p^f)$  and  $\theta_2^s = \theta_2^s(p^f)$ . If  $p^f$  is a bifurcation, due to non-degeneracy both maxima can be locally extended (see Figure 3). We assume that the values at these maxima moves with different speed with respect to the parameter  $p^f$ . Otherwise, the bifurcation value is called *degenerate*. Denote  $a_{min} < a_{max}$  the end points of  $\Gamma$  parametrized by  $p^f$ .

Generic conditions [G0]-[G2] that are introduced below and define  $\mathcal{U} \subset S_r$  are similar to the conditions [C1]-[C3] given by Mather [56]. These conditions can be defined as follows:

Each value  $p^f \in [a_{min}, a_{max}]$  is either regular or bifurcation. Note that the non-degeneracy condition implies that there are at most finitely many bifurcation points. Let  $a_1 < \dots < a_{s-1}$  be the set of bifurcation points in the interval  $(a_{min}, a_{max})$ , and consider the partition of the interval  $[a_{min}, a_{max}]$  by  $\{[a_i, a_{i+1}]\}_{i=0}^{s-1}$ . Here we give an explicit quantitative version of the above condition. Let  $\lambda > 0$  be a parameter.

[G0] There are smooth functions  $\theta_i^s : [a_i - \lambda, a_{i+1} + \lambda] \rightarrow \mathbb{T}$ ,  $i = 0, \dots, s-1$  (see Figure 3), such that for each  $p^f \in [a_i - \lambda, a_{i+1} + \lambda]$ ,  $\theta_i^s(p^f)$  is a local maximum of  $Z(\theta^s, p_*^s(p^f))$  satisfying

$$\lambda I \leq -\partial_{\theta^s \theta^s}^2 Z(\theta_i^s, p) \leq I,$$

where  $I$  is the identity matrix and the inequality is in the sense of quadratic forms.

[G1] For  $p^f \in (a_i, a_{i+1})$ ,  $\theta_i^s$  is the unique maximum for  $Z$ . For  $p^f = a_{i+1}$ ,  $\theta_i^s$  and  $\theta_{i+1}^s$  are the only maxima.

[G2] At  $p^f = a_{i+1}$  the maximum values of  $Z$  have different derivatives with respect to  $p^f$ , i.e.

$$\frac{d}{dp^f} Z(\theta_i^s(a_{i+1}), p_*^s(p^f)) \neq \frac{d}{dp^f} Z(\theta_{i+1}^s(a_{i+1}), p_*^s(p^f)).$$

The conditions at single resonances is that [G0]-[G2] are satisfied for some  $\lambda > 0$ .

Now we are ready to present

*the decomposition into single and (strong) double resonances.*

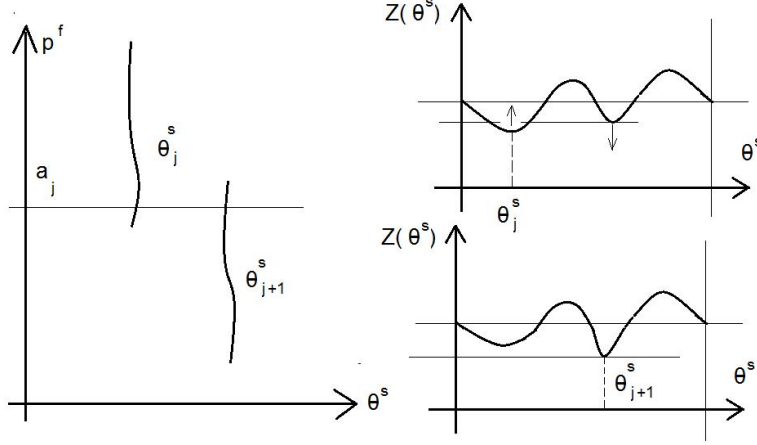


Figure 3: Bifurcations

Following [13] we determine small positive  $\delta = \delta(H_0, \lambda, r, \Gamma^*) < \lambda$  as well as large positive  $C = C(H_0, r, \Gamma^*)$  and  $\bar{E} = \bar{E}(H_0, \lambda, \Gamma^*)$ . Let  $K$  be a positive integer satisfying  $K > C\delta^{4/(r-3)}$  (cf. Remark 3.1 [13]) and such that every  $\Gamma_j \cap \Gamma_{j+1}, j = 1, \dots, N-1$  can be represented as a double resonance. Define  $|\vec{k}|$  as the sum of absolute values of components. Denote

$$\Sigma_K = \{p \in \Gamma \cap B : \exists \vec{k}' = (\vec{k}'_1, k'_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z},$$

$$\vec{k}' \nparallel \vec{k}, \quad |\vec{k}'_1|, |k'_0| \leq K, \quad \vec{k}' \cdot (\partial_p H_0, 1) = 0\}.$$

Call the elements of  $\Sigma_K \subset \Gamma$  *punctures or strong double resonances*. Exclude a neighborhood of each puncture from  $\Gamma$ . Let  $U_{\bar{E}\sqrt{\varepsilon}}(\Sigma_K)$  denote the  $\bar{E}\sqrt{\varepsilon}$ -neighborhood of  $\Sigma_K$ , then  $\Gamma_j \setminus U_{\bar{E}\sqrt{\varepsilon}}(\Sigma_K)$  is a collection of disjoint segments. Each of these segments is called a *passage segment*. By definition crossings of  $\Gamma_j \cap \Gamma_{j+1}$  for each  $j = 1, \dots, N-1$  are (strong) double resonances. In what follows we often omit “strong” for brevity.

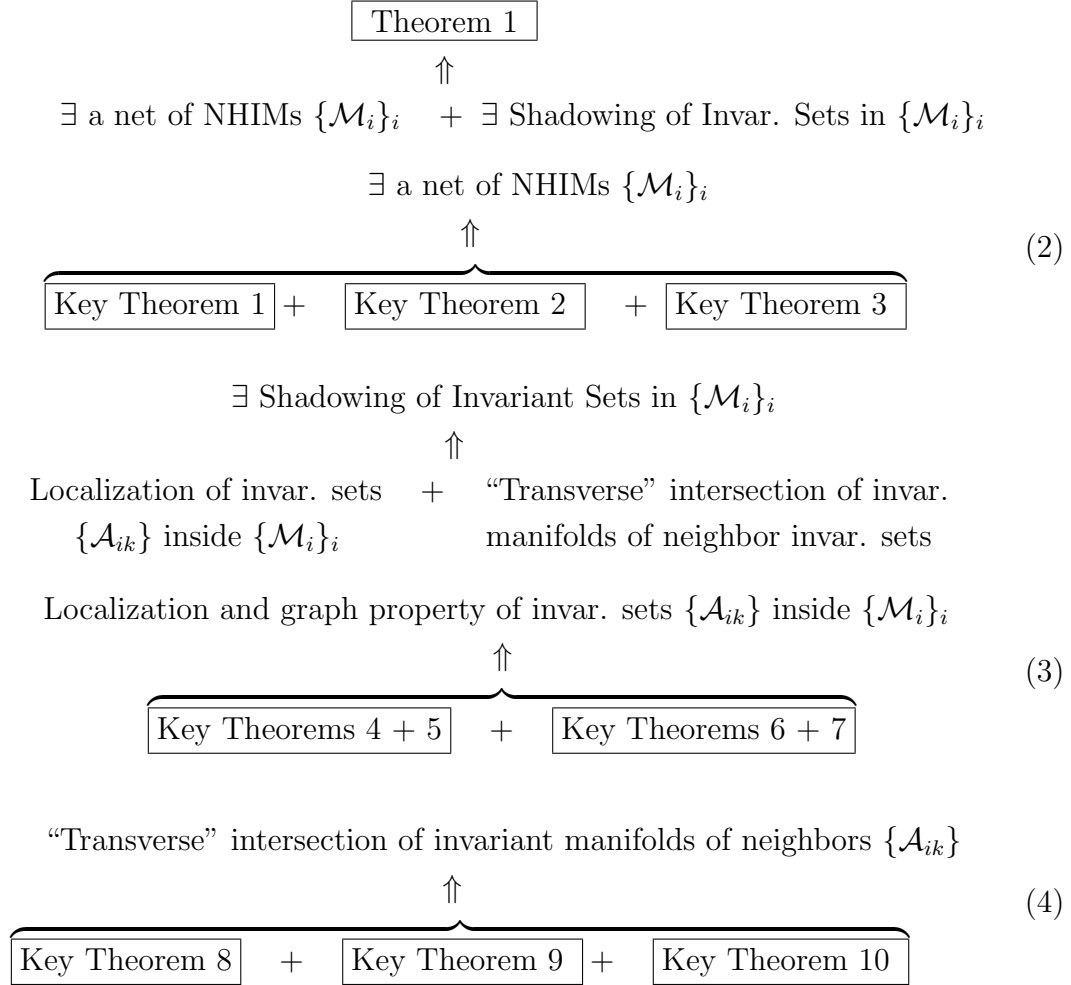
Recall that  $\mathcal{S}^r$  is the unit sphere of  $C^r$ -functions. Let  $\mathcal{U}_{SR}^\lambda$  denote the set of functions in  $\mathcal{S}^r$  such that the conditions [G0]-[G2] are satisfied. Let  $\mathcal{U}_{SR} = \bigcup_{\lambda>0} \mathcal{U}_{SR}^\lambda$ .

**Theorem 3.** *Each  $\mathcal{U}_{SR}^\lambda$  is a  $C^r$ -open, and  $\mathcal{U}_{SR}$  is  $C^r$ -open and  $C^r$ -dense in  $\mathcal{S}^r$ .*

**Remark 2.1.** *We will impose two more sets of non-degeneracy hypothesis at double resonances. The parameter  $K$ , and hence the designation of double resonances depends on  $\lambda$ . Formally, for each  $\lambda > 0$ , our hypothesis at double resonance determines an open dense subset  $\mathcal{U}^\lambda$  of  $\mathcal{U}_{SR}^\lambda$ , and  $\mathcal{U} = \bigcup_{\lambda>0} \mathcal{U}^\lambda$  is open dense by Theorem 3.*

It turns out that the analysis of single and double resonances is *drastically* different and requires different tools.

Before we sink into description of steps of the proof we formally divide it into ten key Theorems. This allows us to partition the proof into smaller (non-equal) parts. We first state these Theorems and derive the Main Theorem. Only after that we proceed with proofs of them<sup>3</sup>. We summarize this discussion in the following diagram:



1. Existence of a net normally hyperbolic invariant manifolds.

(a) Key Theorem 1 establishes existence of crumpled normally hyperbolic invariant cylinders near each connected component of  $\Gamma \setminus U_{\bar{E}\sqrt{\varepsilon}}(\Sigma_K)$ . These

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<sup>3</sup>We reserve the name a “Key Theorem” to refer to the global scheme of the proof summarized here. To prove these Key Theorems we shall prove lemmas, propositions, and “local” Theorems.

cylinders are *crumpled* in the sense that in the original phase space  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$  they are represented by certain three dimensional graphs

$$\{(\Theta_*, P_*)(\theta_*, p_*, t), (\theta_*, p_*, t) \in \mathbb{T} \times [a_-, a_+] \times \mathbb{T}\} \subset \mathbb{T}^2 \times B^2 \times \mathbb{T}$$

with  $\frac{\partial \Theta_*}{\partial p_*} \lesssim 1/\sqrt{\varepsilon}$ . See Section 7.1 and Figure 7 for details. It turns out that asymptotically in  $\varepsilon$  this estimate is optimal, i.e.  $\sup \left\| \frac{\partial \Theta_*}{\partial p_*} \right\| \approx 1/\sqrt{\varepsilon}$  (see Remark 7.1 for details).

- (b) Key Theorem 2 establishes existence of normally hyperbolic invariant cylinders near double resonances at *high energy* or away from singularity. Energy at double resonance is energy of the corresponding slow mechanical system defined in Appendix B, Lemma B.3.
- (c) Key Theorem 3 establishes existence of variety of normally hyperbolic invariant manifolds near strong double resonances at *low (near critical)* energy.

## 2. Localization of invariant (Aubry) sets inside of invariant manifolds.

One of fundamental discoveries in Mather theory and weak KAM is a large class of invariant sets often called *Mather*, *Aubry*, *Mañe* sets. These sets are crucial for our construction. It turns out that these sets are naturally parametrized by cohomologies  $c \in H^1(\mathbb{T}^2, \mathbb{R})$  and usually denoted by  $\tilde{\mathcal{M}}(c)$ ,  $\tilde{\mathcal{A}}(c)$ ,  $\tilde{\mathcal{N}}(c)$  respectively. They belong to the phase space  $\mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T}$  and satisfy

$$\tilde{\mathcal{M}}(c) \subset \tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{N}}(c).$$

For example, these families of sets contain KAM tori as subclass. In our case the parameter  $c$  encodes the information about rotation vector of orbits in  $\tilde{\mathcal{N}}(c)$ . More precisely, if  $(\theta, p, t) \in \tilde{\mathcal{N}}(c)$ , then

$$|p - c| \lesssim \sqrt{\varepsilon} \quad \text{and} \quad |\dot{\theta} - \partial_p H_0(c)| \lesssim \sqrt{\varepsilon}.$$

- (a) Key Theorem 4 proves that for carefully chosen  $c$ 's invariant (Aubry) sets  $\tilde{\mathcal{A}}(c)$ 's belong to the corresponding normally hyperbolic invariant cylinders  $\{\mathcal{M}_i\}_i$  in single resonance. Theorem 5 proves that these sets also have Mather's projected graph property. Namely, projection of each Aubry set onto the configuration space  $\mathbb{T}^2$  is one-to-one with Lipschitz inverse. This essentially means that these invariant sets are *like Aubry-Mather sets for the twist maps*.



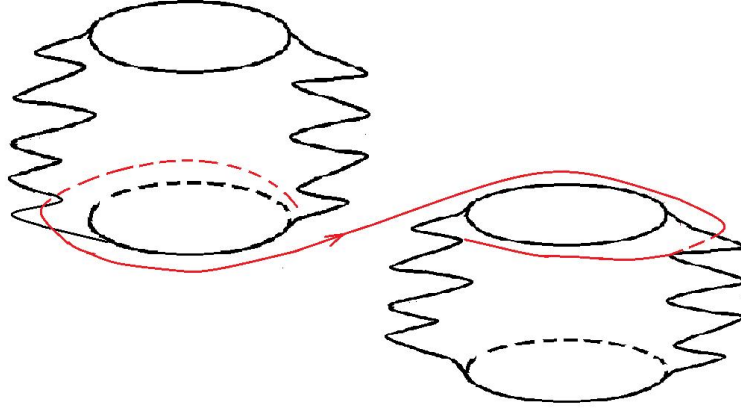


Figure 4: Jump from one cylinder to another in the same homology

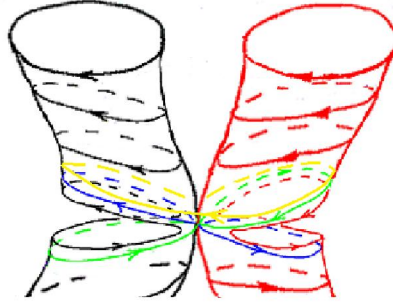


Figure 5: Jump from one cylinder to another in different homologies

- (b) Key Theorem 6 similarly to Key Theorem 4 proves that for carefully chosen  $c$ 's the corresponding invariant (Aubry) sets  $\tilde{\mathcal{A}}(c)$ 's belong to the corresponding normally hyperbolic<sup>4</sup> invariant manifolds  $\{\mathcal{C}_i\}_i$  in a double resonance. Key Theorem 7 proves that these sets also have Mather's projected graph property, described above.
- 3. "Transverse" intersection of invariant manifolds of neighboring invariant Aubry sets to produce shadowing
  - (a) Key Theorem 8 consists of two parts. First, it proves existence of shadowing along crumpled normally hyperbolic invariant cylinders along single resonances. This part essentially follows from Theorem 0.11 [9]. It can

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<sup>4</sup>As a matter of fact there manifolds have a boundary and have only weak invariance, namely, the vector field is tangent

also be proven using the method from [23, 24]. Both methods are inspired by the papers of Mather [53, 54, 56]. The second part establishes existence of an orbit “jumping” from one cylinder to another at bifurcation points.

- (b) Key Theorem 9 proves similar statement to the previous one for double resonances. In particular, for certain (simple) homology directions we prove existence of orbits crossing the double resonance (see curves crossing the origin on Figures 12 and 17 for an heuristic description).
- (c) Key Theorem 10 proves existence of an orbit “jumping” from one cylinder to another. This allows to change from one resonant line to another. This is a crucial element of crossing a strong double resonance.

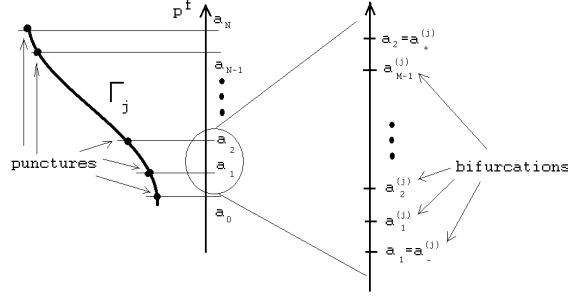


Figure 6: Partition

### 3 Theorems on existence of normally hyperbolic invariant manifolds

Consider the natural projection  $\pi : \mathbb{T}^2 \times B^2 \times \mathbb{T} \longrightarrow B^2$ . A net of normally hyperbolic invariant manifolds has the following meaning. We divide  $\Gamma^*$  into  $\Gamma_j$ 's and each  $\Gamma_j$  into two parts:  $\bar{E}\sqrt{\varepsilon}$ -neighborhoods of all strong double resonances and  $\bar{E}\sqrt{\varepsilon}$ -neighborhood of the complement, i.e

$$\bigcup_{j=1}^N U_{\bar{E}\sqrt{\varepsilon}}(\Sigma_{K,j}) \cup_{j=1}^{N-1} U_{\bar{E}\sqrt{\varepsilon}}(\Gamma_j \cup \Gamma_{j+1}),$$

where  $\Sigma_{K,j}$  are defined above punctures in  $\Gamma_j$ . The complement is called *single resonance*.

#### 3.1 Description of single resonances

Fix some  $j \in \{1, \dots, N\}$ . Consider a passage segment  $[a_-^{(j)}, a_+^{(j)}]$  of a single resonance segment  $\Gamma_j$  which by definition is an interval whose end point is either  $\bar{E}\sqrt{\varepsilon}$  near a puncture or an end point of  $\Gamma_i$ . Let  $\{a_i^{(j)}\}_{i=1}^M \subset [a_-^{(j)}, a_+^{(j)}]$  be an ordered set of bifurcation points with  $a_0^{(j)} = a_-^{(j)}$  and  $a_M^{(j)} = a_+^{(j)}$  being the end points and all others being bifurcation points<sup>5</sup>. This leads to a partition  $[a_-^{(j)}, a_+^{(j)}] = \bigcup_{i=0}^M [a_i^{(j)}, a_{i+1}^{(j)}] \subset \Gamma_j$  (see Figure 6).

Recall that in non-degeneracy conditions [G0]-[G2] there is a parameter  $\lambda$ , which has two different meanings. One is quantitative non-degeneracy of local maxima of averaged potential  $Z(\theta^s, p)$  and the other is extendability of local minima beyond bifurcation points  $[a_i - \lambda, a_{i+1} + \lambda]$ . It is convenient to assign this dependence to different parameters. We denote by  $\delta$  a new (extendability) parameter  $0 < \delta < \lambda$ . For  $\delta = \delta(H_0, H_1, r, \Gamma^*) > 0$  in Theorem 4.1, [13] we prove that “over” each non-boundary

<sup>5</sup> $M$  does depend on  $j$ , but dependence is omitted to keep notations simple

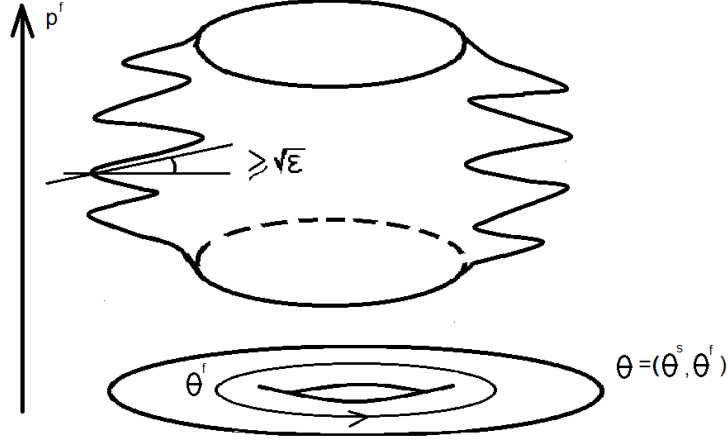


Figure 7: Crumpled Cylinders

segment  $[a_i^{(j)} - \delta, a_{i+1}^{(j)} + \delta] \subset \Gamma_j$  there is a 3-dimensional normally hyperbolic (weakly) invariant cylinder  $\mathcal{C}_i^{(j)}$ . For the boundary segments  $[a_0^{(j)}, a_1^{(j)} + \delta]$  and  $[a_{M-1}^{(j)} - \delta, a_M^{(j)}]$  the corresponding result is Theorem 15. Meaning of “over” is that its projection onto the action space satisfies

$$\text{dist}(\pi(\mathcal{C}_i^{(j)}), [a_i^{(j)} - \delta, a_{i+1}^{(j)} + \delta]) \lesssim \sqrt{\varepsilon},$$

where *dist* is the Hausdorff distance. The same statement holds for boundary segments  $[a_0^{(j)}, a_1^{(j)} + \delta]$  and  $[a_{M-1}^{(j)} - \delta, a_M^{(j)}]$ . Here  $[a_i^{(j)}, a_{i+1}^{(j)}]$  is viewed as the corresponding subset of  $\Gamma_j$  using parametrization by  $p^f$ . Our main result about existence and location of normally hyperbolic invariant cylinders near a single resonance is as follows:

Denote by  $[a_-^i, a_+^i]$  either a non-boundary segment  $[a_i^{(j)} - \delta, a_{i+1}^{(j)} + \delta]$ ,  $i \in \{1, \dots, M-1\}$  or one of boundary segments  $[a_0^{(j)} - 3\sqrt{\varepsilon}, a_1^{(j)} + \delta]$  and  $[a_{M-1}^{(j)} - \delta, a_M^{(j)} + 3\sqrt{\varepsilon}]$ . We use  $O(\cdot)$  to denote a constant independent of  $\varepsilon, \lambda, \delta, r$ , but dependent of  $H_0, H_1, \Gamma_j$ , i.e.  $f = O(g)$  means  $|f| \leq Cg$ . Consider the Hamiltonian  $H_\varepsilon$  in the normal form

$$H_\varepsilon \circ \Phi_\varepsilon = H_0(p) + \varepsilon Z(\theta^s, p) + \varepsilon R(\theta, p, t),$$

where  $\|R\|_{C^2} \leq \delta$  in the region of interest, i.e.  $p$  being  $O(\sqrt{\varepsilon})$ -close to  $(p_*^s(p^f), p^f)$  with  $p^f \in [a_-^i, a_+^i]$  (see notations before (1) and in section 7).

**Key Theorem 1.** *With the above notations for  $\delta$  and  $\varepsilon$  positive and small enough  $\varepsilon > 0$  there exists a  $C^1$  map*

$$(\Theta^s, P^s)(\theta^f, p^f, t) : \mathbb{T} \times [a_-^i, a_+^i] \times \mathbb{T} \longrightarrow \mathbb{T} \times \mathbb{R}$$

such that the cylinder

$$\mathcal{C}_i^{(j)} = \{(\theta^s, p^s) = (\Theta_i^s, P_i^s)(\theta^f, p^f, t) : p^f \in [a_-^i, a_+^i], (\theta^f, t) \in \mathbb{T} \times \mathbb{T}\}$$

is weakly invariant with respect to  $H_\varepsilon$  in the sense that the Hamiltonian vector field is tangent to  $\mathcal{C}_j^{(j)}$ . The cylinder  $\mathcal{C}_i^{(j)}$  is contained in the set

$$V_i^{(j)} := \{(\theta, p, t) : p^f \in [a_-^i, a_+^i], \|\theta^s - \theta_i^s(p^f)\| \leq O(\lambda), \|p^s\| \leq O(\lambda^{5/4} \sqrt{\varepsilon})\},$$

and it contains all the full orbits of  $H_\varepsilon$  contained in  $V_i^{(j)}$ <sup>6</sup>. We have the following estimates

$$\begin{aligned} \|\Theta_j^s(\theta^f, p^f, t) - \theta_*^s(p^f)\| &\leq O(\lambda^{-1}\delta), \\ \|P_j^s(\theta^f, p^f, t)\| &\leq \sqrt{\varepsilon} O(\lambda^{-3/4}\delta), \\ \left\| \frac{\partial \Theta^s}{\partial p^f} \right\| &\leq O\left(\frac{\lambda^{-5/4}\sqrt{\delta}}{\sqrt{\varepsilon}}\right), \quad \left\| \frac{\partial \Theta^s}{\partial(\theta^f, t)} \right\| \leq O(\lambda^{-5/4}\sqrt{\delta}). \end{aligned}$$

Notice that the segment  $[a_-^i, a_+^i]$  can have end points of three types:  $a_\pm^i$  corresponds to the boundary of  $B^2$ , a bifurcation point, and belongs to the  $\bar{E}\sqrt{\varepsilon}$ -boundary of a (strong) double resonance. This theorem for segments whose boundaries do not end at a double resonance is a simplified version of Theorem 4.1 [13]. For a boundary point on ending at a (strong) double resonances follows from Theorem 15. This Theorem is an extension of Theorem 4.1 [13] and its proof is a modification of the proof of the latter Theorem. It turns out that Theorem 4.1 [13] shows existence of a normally hyperbolic weakly invariant cylinder only  $O(\varepsilon^{1/4})$ -away from a strong double resonance. Theorem 15 extends it validity into  $O(\sqrt{\varepsilon})$ -neighborhood.

**Remark 3.1.** The estimates on  $\Theta^s$  and  $P^s$  provides important information about the geometry of the cylinder  $\mathcal{C}_i^{(j)}$ .

The first estimate shows that the  $\Theta^s$ -component (i.e. slow angular component) is localized  $O(\lambda^{-1}\delta)$ -near the global maximum  $\theta_i^s$ .

The second estimate provides localization of  $P^s$  near the origin.

The third and forth estimates show how  $\Theta^s$  depends on  $p^f$  and angular variables  $(\theta^f, t)$ . In Remark 7.1 we justify that generically  $O(\sqrt{\varepsilon})$ -near double resonances  $\Gamma_j \cap \Gamma_{\vec{k}'}$  of a fixed order, e.g.  $|\vec{k}'| \in [K, 2K]$ , we have

$$\max \left\| \frac{\partial \Theta^s}{\partial p^f} \right\| \gtrsim \frac{1}{\sqrt{\varepsilon}},$$

---

<sup>6</sup> Notice that there are orbits exiting from  $V_i^{(j)}$  through the “top” or “bottom”  $p^f = a_\pm^i$ . This prevents up from saying that  $\mathcal{C}_i^{(j)}$  is invariant.

where  $\gtrsim$  means that for some constant depending on  $\delta, \lambda, r, H_0, H_1, \Gamma_j, |\vec{k}'|$ , but independent of  $\varepsilon$ .

This explosion of the upper bound  $\frac{\partial \Theta^s}{\partial p^f}$  exhibits sensitive dependence on “vertical” variable  $p^f$  and is the reason we call there cylinders crumpled (see Figure 7). An upper bound on  $\frac{\partial \Theta^s}{\partial p^f}$  will be essentially used in the proof of Mather’s projected graph theorem (Key Theorem 5).

Dependence on angular  $(\theta^f, t)$ -variables is  $O(\lambda^{-5/4}\delta)$ -small.

In what follows we choose  $\delta$  and  $\varepsilon$  small enough compare to  $\lambda$ . See Theorem 4.1, [13] and Theorem 15 for explicit bounds.

This result shows that normally hyperbolic invariant manifolds connect to  $\bar{E}\sqrt{\varepsilon}$ -neighborhood of strong double resonances.

### 3.2 Description of double resonances and generic properties at high energy

Now we describe an heuristic picture of the  $\bar{E}\sqrt{\varepsilon}$ -neighborhood of a strong double resonance  $p_0$ , i.e.  $p_0$  either belongs to a puncture on  $\Gamma_j$  or to an intersection  $\Gamma_j \cap \Gamma_{j+1}$  for some  $j \in \{1, \dots, N-1\}$ . We note that examples of systems near a double resonance were studied in [16, 43, 45, 46].

We fix two independent resonant lines  $\Gamma = \Gamma_{\vec{k}}$ ,  $\Gamma' = \Gamma_{\vec{k}'}$  for some  $\vec{k} = (\vec{k}_1, k_0)$ ,  $\vec{k}' = (\vec{k}'_1, k'_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$ ,  $\vec{k} \nparallel \vec{k}'$  with  $|\vec{k}|, |\vec{k}'| < K$  and a strong double resonance  $p_0 \in \Gamma \cap \Gamma' \subset B^2$ . This means

$$\vec{k}_1 \cdot (\partial_p H_0(p_0), 1) = 0 \quad , \quad \vec{k}'_1 \cdot (\partial_p H_0(p_0), 1) = 0.$$

Assume that  $\vec{k}$  and  $\vec{k}'$  are space irreducible, i.e.  $\vec{k}_1$  (resp.  $\vec{k}'_1$ ) is either  $(1, 0)$  or  $(0, 1)$ , or  $\gcd(\vec{k}_1) = 1$  (resp.  $\gcd(\vec{k}'_1) = 1$ ). Let  $\Gamma$  be an incoming line, i.e. orbits diffuse toward  $p_0$  along this one (see Figure 12 with  $\Gamma_j = \Gamma$  and  $\Gamma_{j+1} = \Gamma'$ ). Then we define slow angles

$$\varphi^{ss} = \vec{k}_1 \cdot \theta + k_0 t \quad , \quad \varphi^{sf} = \vec{k}'_1 \cdot \theta + k'_0 t.$$

For  $p \in \Gamma \cap \{\|p - p_0\| = \bar{E}\sqrt{\varepsilon}\}$  (see boundary of the ball on Figure 12 crossing  $\Gamma$ ) we have  $\dot{\varphi}^{sf} \gg \dot{\varphi}^{ss}$ . This motivates division into sf — slow-fast and ss — slow-slow. Denote the torus  $\mathbb{T}^s \ni \varphi^s = (\varphi^{ss}, \varphi^{sf})$ . In section B.1, we show that in  $\bar{E}\sqrt{\varepsilon}$ -neighborhood of  $p_0$ , we have the following the normal form

$$H_\varepsilon \circ \Phi(\theta, p, t) = H_0(p) + \varepsilon Z(\vec{k}_1 \cdot \theta + k_0 t, \vec{k}'_1 \cdot \theta + k'_0 t, p) + \varepsilon^{3/2} R(\theta, p, t),$$

where  $Z(\vec{k}_1 \cdot \theta + k_0 t, \vec{k}'_1 \cdot \theta + k'_0 t, p)$  is a proper average of  $H_1$ . In section B.2 we make a linear symplectic coordinate change  $\Phi_L$  by taking

$$p^s = B^T(p - p_0), \quad \text{where} \quad B = \begin{bmatrix} \vec{k}_1 \\ \vec{k}'_1 \end{bmatrix} \quad \text{and} \quad \det B = 1.$$

Since  $\vec{k}$  and  $\vec{k}'$  are irreducible this can be done. After the coordinate changes we have

$$H_\varepsilon \circ \Phi \circ \Phi_L(\varphi^s, p^s, t, E') = \text{const} + K(p^s) - \varepsilon U(\varphi^s) + O(\varepsilon^{3/2}), \quad (5)$$

where

- $K(p^s)$  is a positive definite quadratic form, depending on the Hessian of  $H_0$  at  $p_0$ ,
- $U(\varphi^s)$  is a function, depending on a proper average of  $H_1$ , and  $\text{const}$  is a constant independent of  $\varphi^s$  and  $p^s$ .

We call  $K$  — *the slow kinetic energy* and  $U$  — *the slow potential energy*. They are formally defined in (54). Without loss of generality assume that  $U(\varphi^s) \geq 0$ ,  $U(0) = 0$  and 0 is the only global minimum (see condition [A1]). Denote  $I^s = \sqrt{\varepsilon} p^s$  and call

$$H^s(I^s, \varphi^s) = K(I^s) - U(\varphi^s)$$

*the slow mechanical system*<sup>7</sup>. Its energy is called *slow energy*. In-depth study of this system is the *focal point* of analysis at a strong double resonance. Denote by

$$\mathcal{S}_E = \{(\varphi^s, I^s) : H^s = E\}$$

an energy surface of the mechanical system. According to the Mapertuis principle for a positive “non-critical” energy  $E > U(0) = 0$  orbits of  $H^s$  restricted to  $\mathcal{S}_E$  are reparametrized geodesics of the Jacobi metric

$$g_E(\varphi^s) = 2(E + U(\varphi^s)) K. \quad (6)$$

Fix an integer homology class  $h \in H_1(\mathbb{T}^s, \mathbb{Z})$ . Denote by  $\gamma_h^E$  a shortest closed geodesic of  $g_E$ . Here is the first set of non-degeneracy hypothesis. It concerns with what we call *high slow energy*. This set of genericity hypothesis is the same as in [53]. Later, however, we impose additional hypothesis for low energy and they are different from [53].

Let  $E_0 = E_0(H_0, H_1) > 0$  be small and specified later.

---

<sup>7</sup>Its orbits are time rescaling of orbits of truncation of (5)

[DR1] For each  $E \in [E_0, \bar{E}]$ , each shortest closed geodesic  $\gamma_h^E$  of  $g_E$  in the homology class  $h$  is non-degenerate in the sense of Morse, i.e. the corresponding periodic orbit is hyperbolic.

[DR2] For each  $E > E_0$ , there are at most two shortest closed geodesics of  $g_E$  in the homology class  $h$ .

Let  $E^* > E_0$  be such that there are two shortest geodesics  $\gamma_h^{E^*}$  and  $\bar{\gamma}_h^{E^*}$  of  $\rho_{E^*}$  in the homology class  $h$ . Due to non-degeneracy [DR1] there is a local continuation of  $\gamma_h^{E^*}$  and  $\bar{\gamma}_h^{E^*}$  to locally shortest geodesics  $\gamma_h^E$  and  $\bar{\gamma}_h^E$  for  $E$  near  $E^*$ . For a smooth closed curve  $\gamma$  denote by  $\ell_E(\gamma)$  its  $g_E$ -length.

[DR3] Suppose

$$\frac{d(\ell_E(\gamma_h^E))}{dE}\Big|_{E=E^*} \neq \frac{d(\ell_E(\bar{\gamma}_h^E))}{dE}\Big|_{E=E^*}.$$

This means that the  $g_E$ -lengths of periodic orbits  $\gamma_h^E$  and  $\bar{\gamma}_h^E$  as function of  $E$  have different derivatives at  $E = E^*$ . As a corollary we have only finitely many  $E$ 's with two shortest geodesics.

**Theorem 4.** *Let  $K(p^s)$  be a positive definite quadratic form. Then the set  $\mathcal{U}_{DR}^E$  of functions  $H_1 \in \mathcal{S}^r$  such that for the corresponding averaged potential  $U(\varphi^s)$  the slow mechanical system  $H^s(\varphi^s, p^s) = K(p^s) - U(\varphi^s)$  satisfies conditions [DR1]-[DR3] is  $C^r$ -open and  $C^r$ -dense, where  $r \geq 4$ .*

Consider a partition of energy interval  $[E_0, \bar{E}]$ , which is similar to the partition of a single resonance line (see Figure ??). It follows from condition [DR3] that there are only finitely many values  $\{E_j\}_{j=1}^N \subset [E_0, \bar{E}]$ , where there are exactly two minimal geodesics  $\gamma_h^E$  and  $\bar{\gamma}_h^E$ . Call such  $E$ 's *bifurcation energy values*. Other energy value are called regular. Order bifurcation values:

$$E_0 < E_1 < E_2 < \dots < E_N < \bar{E}.$$

Recall that we denote  $\delta = \delta(H_0, H_1, r, \Gamma^*) > 0$  a small number. In particular, it is chosen such that for any  $j = 1, \dots, N-1$  and  $E \in [E_j, E_{j+1}]$  the unique shortest geodesic  $\gamma_h^E$  has a unique smooth local continuation  $\gamma_h^E$  for  $E \in [E_j - \delta, E_{j+1} + \delta]$ . Consider the union

$$\mathcal{M}_h^{E_j, E_{j+1}} = \cup_{E \in [E_j - \delta, E_{j+1} + \delta]} \gamma_h^E.$$

For the boundary intervals we take union over  $[E_0, E_1 + \delta]$  and  $[E_N - \delta, \bar{E} + \delta]$ . It follows from Morse non-degeneracy of  $\gamma_h^E$  that  $\mathcal{M}_h^{E_j, E_{j+1}}$  is a NHIC with the boundary.

We will see in Corollary B.5 that a rescaling of (5)  $C^1$ -converges to  $H^s$  as Hamiltonian vector fields. Using standard persistence of normally hyperbolic invariant cylinders, we obtain the following statement.



**Key Theorem 2.** *For each  $j = 1, \dots, N$  and small  $\varepsilon > 0$  the Hamiltonian  $H_\varepsilon$  has a normally hyperbolic weakly invariant manifold  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$ , i.e. the Hamiltonian vector field of  $H_\varepsilon$  is tangent to  $\mathcal{M}_{h,\varepsilon}^j$ . Moreover, the intersection of  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$  with the regions  $\{E_j - \delta \leq H^s \leq E_{j+1} + \delta\} \times \mathbb{T}$  is a graph over  $\mathcal{M}_h^{E_j, E_{j+1}}$ .*

*The same holds for the boundary intervals  $[E_0, E_1 + \delta]$  and  $[E_N - \delta, \bar{E} + \delta]$ .*

Construction of normally hyperbolic weakly invariant manifolds in the high energy region  $E \in (E_0, \bar{E})$  is somewhat similar to the one in single resonance. All such invariant manifolds after projection onto action space are located in  $O(\sqrt{\varepsilon})$ -neighborhood of  $p_0$ .

### 3.3 Description of double resonances (low energy)

Now we turn to the low (near critical) energy region  $U_{E_0\sqrt{\varepsilon}}(p_0)$ . The slow mechanical system  $H^s = K - U$  has at least two special resonant lines  $\Gamma_1^s$  and  $\Gamma_2^s$  such that “over” each one there is a 3-dimensional normally hyperbolic weakly invariant manifold  $\mathcal{M}_{h_1,\varepsilon}^{E_0,s}$  and  $\mathcal{M}_{h_2,\varepsilon}^{E_0,s}$ . The important feature of these manifolds is that they “go through” the strong double resonance  $p_0$  as shown on Figure 12. We call these manifolds simple loop manifolds by the reason explained below (see also [47]).

It turns out that these and other normally hyperbolic invariant manifolds have *kissing* property near  $p_0$ , which is a crucial element of “jumping” from one manifold to another. Now we give a formal description.

Recall that  $p_0 \in \Gamma \cap \Gamma'$ , where  $\Gamma$  and  $\Gamma'$  are two resonant lines. Naturally,  $\Gamma$  induces an integer homology class  $h \in H^1(\mathbb{T}^s, \mathbb{Z})$ . Recall that  $0 \in \mathbb{T}^s \times \mathbb{R}^s \simeq \mathbb{T}^2 \times \mathbb{R}^2$  is the singular point of the Jacobi metric  $\rho_0$ . It is a saddle fixed point for the slow Hamiltonian  $H^s$ . Let  $\gamma_h^0$  be a shortest geodesic in the homology  $h$  with respect to the Jacobi metric  $g_0$ . Mather [61] proved that generically we have the following cases:

- Definition 3.1.**
1.  $0 \in \gamma_h^0$  and  $\gamma_h^0$  is not self-intersecting. Call such homology class  $h$  simple non-critical and the corresponding geodesic  $\gamma_h^0$  simple loop.
  2.  $0 \in \gamma_h^0$  and  $\gamma_h^0$  is self-intersecting. Call such homology class  $h$  non-simple and the corresponding geodesic  $\gamma_h^0$  non-simple.
  3.  $0 \notin \gamma_h^0$ , then  $\gamma_h^0$  is a regular geodesic. Call such homology class  $h$  simple critical.

It turns out there are open sets of mechanical systems, where each of the three cases occurs. It is only the second item which is somewhat unusual. Notice that  $\gamma_h^0$  is self-intersecting *only if*  $0 \in \gamma_h^0$ , otherwise, curve shortening argument applies at the intersection.

We point out that there are always at least two simple homology classes  $h_1, h_2 \in H^1(\mathbb{T}^s, \mathbb{Z})$ . To prove that minimize over all integer homology classes  $h' \in H^1(\mathbb{T}^s, \mathbb{Z})$  and geodesics passing through the origin. Pick two minimal shortest  $h_1$  and  $h_2$ . The corresponding minimal geodesics  $\gamma_{h_i}^0$ ,  $i = 1, 2$  are non-self-intersecting<sup>8</sup>.

Notice that the condition  $0 \in \gamma_h^0$  for a geodesic corresponds to a homoclinic orbit to the origin in the phase space. They are given by intersections of stable and unstable manifolds of the saddle 0. Existence of transverse intersections implies that there are open sets of mechanical systems with  $\gamma_h^0$  being self-intersecting for some  $h \in H^1(\mathbb{T}^s, \mathbb{Z})$  and, therefore, passing through the origin.

If minimal geodesics of  $\rho_0$  are self-intersecting the situation was described by Mather [57].

**Lemma 3.2.** *Let  $h$  be a non-simple homology class. Then generically  $\gamma_h^0$  is the concatenation of two simple loops, possibly with multiplicities. More precisely, given  $h \in H_1(\mathbb{T}^s, \mathbb{Z})$  generically there are homology classes  $h_1, h_2 \in H_1(\mathbb{T}^s, \mathbb{Z})$  and integers  $n_1, n_2 \in \mathbb{Z}_+$  such that the corresponding minimal geodesics  $\gamma_{h_1}^0$  and  $\gamma_{h_2}^0$  are simple and  $h = n_1 h_1 + n_2 h_2$ .*

We impose the following assumption

[A0] Assume that for the Hamiltonian  $H^s$  and the homology class  $h$ , there is a unique shortest curve  $\gamma_h^0$  for the critical Jacobi metric with homology  $h$ . If  $h$  is non-simple, then it is the concatenation of two simple loops, possibly with multiplicities.

For  $E > 0$ ,  $\gamma_h^E$  has no self intersection. As a consequence, there is a unique way to represent  $\gamma_h^0$  as a concatenation of  $\gamma_{h_1}^0$  and  $\gamma_{h_2}^0$ . Denote  $n = n_1 + n_2$ , we have the following

**Lemma 3.3.** *There exists a sequence  $\sigma = (\sigma_1, \dots, \sigma_n) \in \{1, 2\}^n$ , unique up to cyclical translation, such that*

$$\gamma_h^0 = \gamma_{h_{\sigma_1}}^0 * \dots * \gamma_{h_{\sigma_n}}^0.$$

Now we add more assumptions to [A0] and describe generic properties of the Jacobi metric. We emphasize that *analysis of the phase space, not of the configuration space*, provides additional valuable information! This is a crucial difference with Mather [54, 57].

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<sup>8</sup>otherwise,  $\gamma_{h_i}^0$  can be decomposed into a sum of at least two other geodesics crossing at the origin and, therefore, is not shortest.

### 3.4 Generic properties of homoclinic orbits and genericity at low energy

Pick a simple critical homology class  $h \in H^1(\mathbb{T}^s, \mathbb{Z})$ . Recall that for  $0 \in \gamma_h^0$  in the phase space it corresponds to a homoclinic orbit to the origin. Now we make the following assumptions:

- [A1] The potential  $U$  has a unique non-degenerate minimum at 0 and  $U(0) = 0$ .
- [A2] The linearization of the Hamiltonian flow at  $(0, 0)$  has distinct eigenvalues  $-\lambda_2 < -\lambda_1 < 0 < \lambda_1 < \lambda_2$

In a neighborhood of  $(0, 0)$ , there exist a smooth local system of coordinates  $(u_1, u_2, s_1, s_2) = (u, s)$  such that the  $u_i$ -axes correspond to the eigendirections of  $\lambda_i$  and the  $s_i$ -axes correspond to the eigendirections of  $-\lambda_i$  for  $i = 1, 2$ . Let  $\gamma^+ = \gamma_{h,+}^0$  be a homoclinic orbit of  $(0, 0)$  under the Hamiltonian flow of  $H^s$ . Denote  $\gamma^- = \gamma_{h,-}^0$  the time reversal of  $\gamma_{h,+}^0$ , which is the image of  $\gamma_{h,+}^0$  under the involution  $I^s \mapsto -I^s$  and  $t \mapsto -t$ . We call  $\gamma^+$  (resp.  $\gamma^-$ ) *simple loop*.

We assume the following of the homoclinics  $\gamma^+$  and  $\gamma^-$ .

- [A3] The homoclinics  $\gamma^+$  and  $\gamma^-$  are not tangent to  $u_2$ -axis or  $s_2$ -axis at  $(0, 0)$ . This, in particular, imply that the curves are tangent to the  $u_1$  and  $s_1$  directions. We assume that  $\gamma^+$  approaches  $(0, 0)$  along  $s_1 > 0$  in the forward time, and approaches  $(0, 0)$  along  $u_1 > 0$  in the backward time;  $\gamma^-$  approaches  $(0, 0)$  along  $s_1 < 0$  in the forward time, and approaches  $(0, 0)$  along  $u_1 < 0$  in the backward time (see Figure 9).

For the non-simple case, we consider two homoclinics  $\gamma_1$  and  $\gamma_2$  that are in the same direction instead of being in the opposite direction. More precisely, the following is assumed.

- [A3'] The homoclinics  $\gamma_1$  and  $\gamma_2$  are not tangent to  $u_2$ -axis or  $s_2$ -axis at  $(0, 0)$ . Both  $\gamma_1$  and  $\gamma_2$  approaches  $(0, 0)$  along  $s_1 > 0$  in the forward time, and approaches  $(0, 0)$  along  $u_1 > 0$  in the backward time.

Given  $d > 0$  and  $0 < \delta < d$ , let  $B_d$  be the  $d$ -neighborhood of  $(0, 0)$  and let

$$\Sigma_{\pm}^s = \{s_1 = \pm\delta\} \cap B_d, \quad \Sigma_{\pm}^u = \{u_1 = \pm\delta\} \cap B_d$$

be four local sections contained in  $B_d$ . We have four local maps

$$\Phi_{\text{loc}}^{++} : U^{++}(\subset \Sigma_+^s) \longrightarrow \Sigma_+^u, \quad \Phi_{\text{loc}}^{-+} : U^{-+}(\subset \Sigma_-^s) \longrightarrow \Sigma_+^u,$$

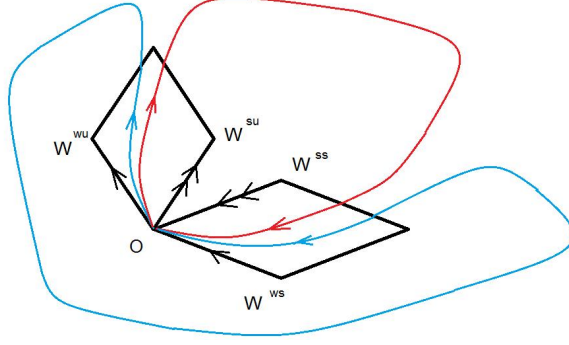


Figure 8: Homoclinic orbits to the origin

$$\Phi_{\text{loc}}^{+-} : U^{+-}(\subset \Sigma_+^s) \longrightarrow \Sigma_-^u, \quad \Phi_{\text{loc}}^{--} : U^{--}(\subset \Sigma_-^s) \longrightarrow \Sigma_-^u.$$

The local maps are defined in the following way. Let  $(u, s)$  be in the domain of one of the local maps. If the orbit of  $(u, s)$  escapes  $B_d$  before reaching the destination section, then the map is considered undefined there. Otherwise, the local map sends  $(u, s)$  to the first intersection of the orbit with the destination section. The local map is not defined on the whole section and its domain of definition will be made precise later.

For the case of simple loop, i.e. assuming [A3], we can define two global maps corresponding to the homoclinics  $\gamma^+$  and  $\gamma^-$ . By assumption [A3], for a sufficiently small  $\delta > 0$ , the homoclinic  $\gamma^+$  intersects the sections  $\Sigma_+^u$  and  $\Sigma_+^s$  and  $\gamma^-$  intersects  $\Sigma_-^u$  and  $\Sigma_-^s$ . Let  $p^+$  and  $q^+$  (resp.  $p^-$  and  $q^-$ ) be the intersection of  $\gamma^+$  (resp.  $\gamma^-$ ) with  $\Sigma_+^u$  and  $\Sigma_+^s$  (resp.  $\Sigma_-^u$  and  $\Sigma_-^s$ ). Smooth dependence on initial conditions implies that for the neighborhoods  $V^\pm \ni q^\pm$  there are a well defined Poincaré return maps

$$\Phi_{\text{glob}}^+ : V^+ \longrightarrow \Sigma_+^s, \quad \Phi_{\text{glob}}^- : V^- \longrightarrow \Sigma_-^s.$$

When [A3'] is assumed, for  $i = 1, 2$ ,  $\gamma^i$  intersect  $\Sigma_+^u$  at  $q^i$  and intersect  $\Sigma_+^s$  at  $p^i$ . The global maps are denoted

$$\Phi_{\text{glob}}^1 : V^1 \longrightarrow \Sigma_+^s, \quad \Phi_{\text{glob}}^2 : V^1 \longrightarrow \Sigma_+^s.$$

The composition of local and global maps for the family of periodic orbits shadowing  $\gamma^+$  is illustrated in Figure 8.

We will assume that the global maps are “in general position”. We will only phrase our assumptions [A4a] and [A4b] for the homoclinic  $\gamma^+$  and  $\gamma^-$ . The assumptions for  $\gamma^1$  and  $\gamma^2$  are identical, only requiring different notations and will be called [A4a'] and [A4b']. Let  $W^s$  and  $W^u$  denote the local stable and unstable manifolds of  $(0, 0)$ . Note that  $W^u \cap \Sigma_\pm^u$  is one-dimensional and contains  $q^\pm$ . Let  $T^{uu}(q^\pm)$  be the tangent

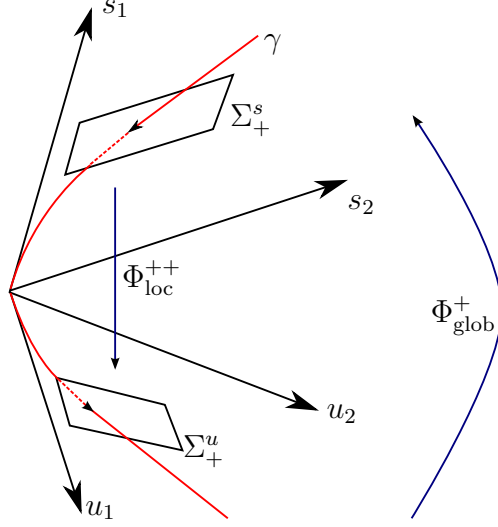


Figure 9: Global and local maps for  $\gamma^+$

direction to this one dimensional curve at  $q^\pm$ . Similarly, we define  $T^{ss}(p^\pm)$  to be the tangent direction to  $W^s \cap \Sigma_\pm^s$  at  $p^\pm$ .

[A4a] Image of strong stable and unstable directions under  $D\Phi_{\text{glob}}^\pm(q^\pm)$  is transverse to strong stable and unstable directions at  $p^\pm$  on the energy surface  $S_0 = \{H^s = 0\}$ . For the restriction to  $S_0$  we have

$$D\Phi_{\text{glob}}^+(q^+)|_{TS_0} T^{uu}(q^+) \pitchfork T^{ss}(p^+), \quad D\Phi_{\text{glob}}^-(q^-)|_{TS_0} T^{uu}(q^-) \pitchfork T^{ss}(p^-).$$

[A4b] Under the global map, the image of the plane  $\{s_2 = u_1 = 0\}$  intersects  $\{s_1 = u_2 = 0\}$  at a one dimensional manifold, and the intersection transversal to the strong stable and unstable direction. More precisely, let

$$L(p^\pm) = D\Phi_{\text{glob}}^\pm(q^\pm)\{s_2 = u_1 = 0\} \cap \{s_1 = u_2 = 0\},$$

we have that  $\dim L(p^\pm) = 1$ ,  $L(p^\pm) \neq T^{ss}(p^\pm)$  and  $D(\Phi_{\text{glob}}^\pm)^{-1}L(p^\pm) \neq T^{uu}(q^\pm)$ .

[A4'] Suppose conditions [A4a] and [A4b] hold for both  $\gamma_1$  and  $\gamma_2$ .

In the case that the homology  $h$  is simple and  $0 \notin \gamma_h^0$ , we assume

[A4''] The closed geodesic  $\gamma_h^0$  is hyperbolic.

We would like to emphasize that these non-degeneracy assumptions are restrictions on the Hamiltonian flow in *the phase space*. Mather [58] imposes non-degeneracy assumptions on the Jacobi metric in *the configuration space*.

**Theorem 5.** *Let  $K(p^s)$  be a positive definite quadratic form. The set  $\mathcal{U}_{DR}^{crit}(p_0)$  of functions  $H_1 \in \mathcal{S}^r$  such that for the corresponding averaged potential  $U(\varphi^s)$  the slow mechanical system  $H^s(\varphi^s, p^s) = K(p^s) - U(\varphi^s)$  satisfies conditions [A0]-[A4] is  $C^r$ -open and  $C^r$ -dense, where  $r \geq 2$ .*

In what follows we need a quantitative version of conditions [A1]-[A4]. Let  $\kappa > 0$ . We say conditions [A1]-[A4] holds with non-degeneracy parameter  $\kappa$  if

- $U''(0) > \kappa$ ,  $\lambda_2 - \lambda_1 > 5\kappa$ ;
- $\gamma^\pm$  crosses  $\Sigma_\pm^u$  and  $\Sigma_\pm^s$  transversally for  $\delta > \kappa$ ;
- For the restriction to the energy surface  $S_0$  we have

$$\angle(D\Phi_{\text{glob}}^+(q^+)|_{TS_0} T^{uu}(q^+), T^{ss}(p^+)) > \kappa,$$

$$\angle(D\Phi_{\text{glob}}^-(q^-)|_{TS_0} T^{uu}(q^-), T^{ss}(p^-)) > \kappa.$$

and

$$\angle(L(p^\pm), T^{ss}(p^\pm)) > \kappa, \quad \angle(D(\Phi_{\text{glob}}^\pm)^{-1}L(p^\pm), T^{uu}(q^\pm)) > \kappa.$$

We show that under our assumptions, for small energy, there exists “shadowing” periodic orbits close to the homoclinics. These orbits were studied by Shil’nikov [70], Shil’nikov-Turaev [71], and Bolotin-Rabinowitz [20].

**Theorem 6.** *1. In the simple loop case, we assume that the conditions A1 - A4 hold for  $\gamma^+$  and  $\gamma^-$ . Then there exists  $E_0 > 0$  such that for each  $0 < E \leq E_0$ , there exists a unique periodic orbit  $\gamma_+^E$  corresponding to a fixed point of the map  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}$  restricted to the energy surface  $\mathcal{S}_E$ .*

*For each  $0 < E \leq E_0$ , there exists a unique periodic orbit  $\gamma_-^E$  corresponding to a fixed point of the map  $\Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{--}$  restricted to the energy surface  $\mathcal{S}_E$ .*

*For each  $-E_0 \leq E < 0$ , there exists a unique periodic orbit  $\gamma_c^E$  corresponding to a fixed point of the map  $\Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{+-} \circ \Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{-+}$  restricted to the energy surface  $\mathcal{S}_E$ .*

*2. In the non-simple case, assume that the assumptions [A1], [A2], [A3'] and [A4'] hold for  $\gamma^1$  and  $\gamma^2$ . Then there exists  $E_0 > 0$  such that for  $0 < E \leq E_0$ , the following hold. For any  $\sigma = (\sigma_1, \dots, \sigma_n) \in \{1, 2\}^n$ , there is a unique periodic orbit, denoted by  $\gamma_\sigma^E$ , corresponding to a unique fixed point of the map*

$$\prod_{i=1}^n (\Phi_{\text{glob}}^{\sigma_i} \circ \Phi_{\text{loc}}^{++})$$

*restricted to the energy surface  $\mathcal{S}_E$ . (Product stands for composition of maps).*

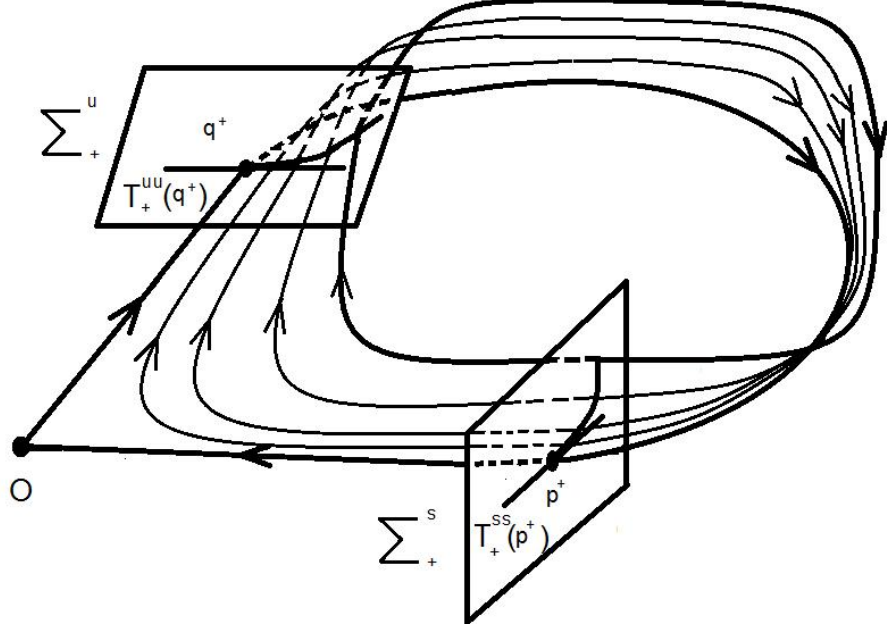


Figure 10: Periodic orbits shadowing  $\gamma^+$

The family of periodic orbits  $\gamma_+^E$  is depicted in Figure 10.

**Theorem 7.** *In the case of simple loop, assume that conditions [A1]-[A4] are satisfied with  $\gamma^+ = \gamma_{h,+}^0$  and  $\gamma^- = \gamma_{h,-}^0$ . For this choice of  $\gamma^+$  and  $\gamma^-$ , let  $\gamma_+^E$ ,  $\gamma_c^E$  and  $\gamma_-^E$  be the family of periodic orbits obtained from part 1 of Theorem 6. Then the set*

$$\mathcal{M}_h^{E_0,s} = \bigcup_{0 < E \leq E_0} \gamma_+^E \cup \gamma^+ \cup \bigcup_{-E_0 \leq E < 0} \gamma_c^E \cup \gamma^- \cup \bigcup_{0 < E \leq E_0} \gamma_-^E$$

is a  $C^1$  smooth normally hyperbolic invariant manifold with boundaries  $\gamma_+^{E_0}$ ,  $\gamma_c^{E_0}$  and  $\gamma_-^{E_0}$  (see Figure 11).

In the case of non-simple loop, assume that [A1], [A2], [A3'] and [A4'] are satisfied with  $\gamma^1 = \gamma_{h_1}^0$  and  $\gamma^2 = \gamma_{h_2}^0$ . Let  $\gamma_\sigma^E$  denote the family of periodic orbits obtained from applying part 2 of Theorem 6 to the sequence  $\sigma$  determined by Lemma 3.3. We have that for any  $0 < e < E_0$ , the set

$$\mathcal{M}_h^{e,E_0} = \bigcup_{e \leq E \leq E_0} \gamma_\sigma^E$$

is a  $C^1$  smooth normally hyperbolic invariant manifold with the boundary (see Figure 11).

Moreover, the conclusions of the theorem also applies to any sufficiently small  $C^2$  perturbation of the slow mechanical system  $H^s = K - U$ , where the  $C^2$ -size of such

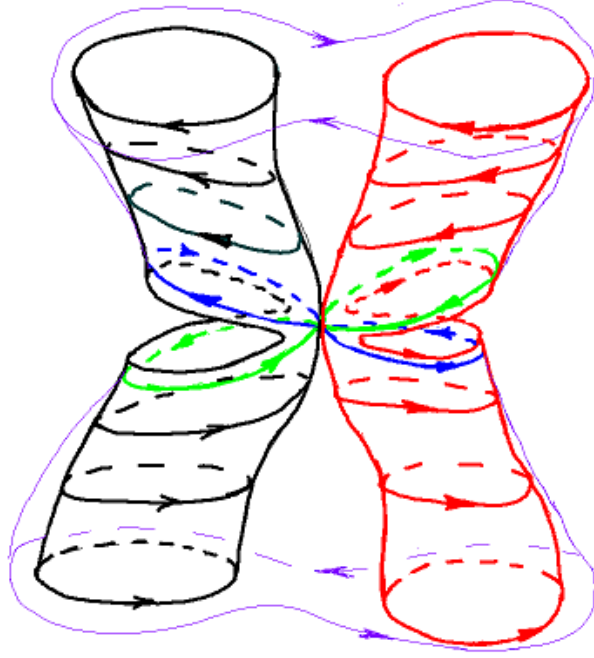


Figure 11: Normally hyperbolic invariant manifolds with kissing property

a perturbation depends only on quantitative non-degeneracy of hypothesis [A1]-[A4], given by  $\kappa > 0$ , and  $C^2$ -norms of  $H_0$  and  $H_1$ .

**Remark 3.2.** Due to hyperbolicity the cylinder  $\mathcal{M}_h^{E_0,s}$  is  $C^\alpha$  for any  $\alpha$  satisfying  $1 < \alpha < \lambda_2/\lambda_1$ .

If  $h_1$  and  $h_2$  corresponds to simple loops, then the corresponding invariant manifolds  $\mathcal{M}_{h_1}^{E_0,s}$  and  $\mathcal{M}_{h_2}^{E_0,s}$  have a tangency along a **two dimensional plane at the origin**. One can say that we have “kissing manifolds”, see Figure 11. This tangency is persistent<sup>9</sup> !

**Remark 3.3.** In the notations of the beginning of this section for small enough  $E_0$  and  $0 \leq E \leq E_0$  we show that

— in the simple loop case, the shadowing orbits  $\gamma_\pm^E$  coincides with the minimal geodesics  $\gamma_{h,\pm}^E$  (see Propositions 8.10 and 8.11).

— in the non-simple case,  $\gamma_{\sigma,\pm}^E$  coincides with  $\gamma_{h,\pm}^E$ . By Lemma 3.3,  $\sigma$  is uniquely determined by  $h$  (see Proposition 8.14 and Remark 8.3).

<sup>9</sup>A. Sorrentino called persistence of this picture “power of love”



**Remark 3.4.** Using Theorem 7 and Remark 3.3, if  $H^s$  satisfies conditions [A0]-[A4], then the shortest geodesic  $\gamma_h^E$  is unique and hyperbolic for any  $E > 0$ . Moreover, this is an open condition, due to the last statement of Theorem 7. This implies the set

$$\mathcal{U}_{DR}^{crit}(p_0) \cap \bigcap_{E_0 > 0} \mathcal{U}_{DR}^{E_0}(p_0)$$

is an open and dense set.

Denote  $\mathcal{U}_{DR}^E = \bigcap_{E_0 > 0} \mathcal{U}_{DR}^{E_0}(p_0)$ , then for each  $p_0$ , the set  $\mathcal{U}_{DR}^{crit}(p_0) \cap \mathcal{U}_{DR}^{E_0}(p_0)$  is open and dense. Since for each  $\lambda > 0$  there are only finitely many double resonances, the set

$$\mathcal{U} = \bigcup_{\lambda > 0} \mathcal{U}_{SR}^\lambda \cap \left( \bigcap_{p_0} \mathcal{U}_{DR}^E(p_0) \cap \mathcal{U}_{DR}^{crit}(p_0) \right),$$

is open and dense.

**Key Theorem 3.** Let  $r \geq 4$  and assume that  $H_\epsilon = H_0 + \epsilon H_1$  is  $C^r$  and such that  $H_1 \in \mathcal{U}_{DR}^{crit}(p_0) \subset \mathcal{S}^r$ , i.e.  $H_\epsilon$  satisfies the conditions of Theorem 7.

1. If homology  $h$  is simple, then  $H_\epsilon$  exhibits normally hyperbolic weakly invariant<sup>10</sup> manifold  $\mathcal{M}_{h,\epsilon}^{E_0,s}$ . Moreover, the intersection of  $\mathcal{M}_{h,\epsilon}^{E_0,s}$  with the regions  $\{-E_0 \leq H^s \leq E_0\} \times \mathbb{T}$  is a  $C^1$ -graph over  $\mathcal{M}_h^{E_0,s}$ .
2. If homology  $h$  is non-simple. Then  $H_\epsilon$  exhibits simple normally hyperbolic weakly<sup>11</sup> invariant cylinders  $\mathcal{M}_{h_1,\epsilon}^{E_0,s}$  and  $\mathcal{M}_{h_2,\epsilon}^{E_0,s}$  satisfying the conditions of the previous item. Moreover, for fixed  $0 < e < E_0$ , there exists a normally hyperbolic weakly invariant cylinder  $\mathcal{M}_{h,\epsilon}^{e,E_0}$  whose intersection with the region  $\{e \leq H^s \leq E_0\} \times \mathbb{T}$  is a  $C^1$ -graph over  $\mathcal{M}_h^{e,E_0}$ .

**Remark 3.5.** Notice that the theorem is stated for the Hamiltonian  $H_\epsilon$ .  $H_\epsilon$  is related to  $H^s$  in Appendix B (sections B.1 and B.2). After a proper normal form  $\Phi$  and a linear transformation  $\Phi'_L$  we have that  $H_\epsilon \circ \Phi \circ \Phi'_L$  is a  $O(\epsilon^{1/2})$ -small fast time-periodic perturbation of  $H^s$ .

Sometimes  $\mathcal{M}_{h,\epsilon}^{e,E_0}$  is referred as a flower cylinder due to its shape (see Figure 11).

These are all the essential stages of building a net of normally hyperbolic invariant manifolds. The next important step is to construct diffusing orbits following this net of invariant manifolds.

<sup>10</sup>as before weakly means that the vector field of the Hamiltonian  $H_\epsilon$  is tangent to  $\mathcal{M}_{h,\epsilon}^{E_0,s}$

<sup>11</sup>“weakly” has the same as before

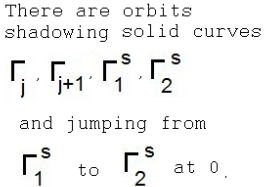


Figure 12: An heuristic picture of a double resonance

### 3.5 Heuristics of the diffusion across double resonances

Before we go into description of sophisticated variational shadowing techniques we present heuristics of crossing a strong double resonance  $p_0 \in \Gamma \cap \Gamma'$ . Recall that  $\Gamma$  is an incoming resonant line and  $h$  the homology class induced by  $\Gamma$  on  $\mathbb{T}^s$  (see Figure 12). As we described above generically there are three cases:

1. the limiting  $\gamma_h^0$  is a simple curve and  $0 \in \gamma_h^0$ ;
2. the limiting  $\gamma_h^0$  is a non-simple curve,  $0 \in \gamma_h^0$ , and there are two simple curves  $\gamma_{h_1}^0 = \gamma_1$  and  $\gamma_{h_2}^0 = \gamma_2$  and two integers  $n_1, n_2 \in \mathbb{Z}_+$  such that  $h = n_1 h_1 + n_2 h_2$ ;
3. the limiting  $\gamma_h^0$  is a simple curve and  $0 \notin \gamma_h^0$ ;

In order to see the cases below one can follow from the incoming arrow to all the outgoing arrows on Figure 12

### 3.5.1 Crossing through along a simple loop $\gamma_h^0 \ni 0$

If homology  $h$  is simple, then an orbit enters along a normally hyperbolic weakly invariant manifold  $\mathcal{M}_{h,\varepsilon}^{E_0,s}$  established by Key Theorems 2 and 3. We show that it can diffuse along  $\mathcal{M}_{h,\varepsilon}^{E_0,s}$  across  $p_0$  to “the other side” (see Figure 12 in homologies and Figure 17, part a) in cohomologies ).

### 3.5.2 Crossing through along a simple loop $\gamma_h^0 \not\equiv 0$

Recall that mechanical systems are invariant under involution:  $I^s \rightarrow -I^s$  and  $t \rightarrow -t$ . If homology  $h$  is simple, then the mechanical system has a normally hyperbolic invariant cylinder  $\mathcal{M}_h^{E_0,s}$  and its involution  $\mathcal{M}_{-h}^{E_0,s}$ . A diffusing orbit enters along a normally hyperbolic weakly invariant cylinder  $\mathcal{M}_{h,\varepsilon}^{E_0,s}$  (which is a perturbation of  $\mathcal{M}_h^{E_0,s}$ ). Then we reach zero energy surface and go through the origin to the “opposite” normally hyperbolic weakly invariant cylinder  $\mathcal{M}_{-h,\varepsilon}^{E_0,s}$  (which is a perturbation of  $\mathcal{M}_{-h}^{E_0,s}$ ) and continues on “the other side”. It corresponds to exactly the same picture as the previous case.

### 3.5.3 Crossing through along a non-simple loop $\gamma_h^0$

If  $h$  is non-simple, i.e the union of two simple loops, then an orbit enters along a normally hyperbolic weakly invariant cylinder  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  reaches a small energy  $e$ . As energy  $E$  decreases to zero the cylinder  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  goes toward the origin and becomes a flower cylinder and its boundary approach the union of simple cylinders  $\mathcal{M}_{h_1,\varepsilon}^{E_0,s}$  and  $\mathcal{M}_{h_2,\varepsilon}^{E_0,s}$  (see Figure 11). For a small enough energy  $e$  and outside of a tiny neighborhood of the origin the flower cylinder  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  is almost tangent to one of simple normally hyperbolic weakly invariant cylinders  $\mathcal{M}_{h_i,\varepsilon}^{E_0,s}$ ,  $i = 1$  or  $2$ . Near the origin both normally hyperbolic weakly invariant manifolds have least contracting and least expanding directions almost parallel to the  $u_1 s_1$ -plane. Moreover, their invariant manifolds have tangent directions which are almost parallel to the  $u_2 s_2$ -plane. As a result there should be transverse intersection of invariant manifolds of  $\mathcal{M}_{h_i,\varepsilon}^{e,E_0}$  and  $\mathcal{M}_{h_i,\varepsilon}^{E_0,s}$ . This implies

*persistent existence of orbits jumping from  
the flower cylinder  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  to a simple loop one  $\mathcal{M}_{h_i,\varepsilon}^{E_0,s}$ .*

Then such orbits can cross the double resonance along  $\mathcal{M}_{h_i,\varepsilon}^{E_0,s}$ . After that it jumps back on the opposite branch of  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  and can diffuse away along it as before (see Figure 12 in homologies and Figure 17, part b) in cohomologies).

### 3.5.4 Turning a corner from $\Gamma$ to $\Gamma'$

Let  $o_\varepsilon \subset \mathbb{T}^2 \times B^2 \times \mathbb{T}$  denote the periodic orbit whose projection onto  $\mathbb{T}^2 \times B^2$  is near the origin.

Now we cross a strong double resonance by entering along  $\Gamma$  and exiting along  $\Gamma'$ . Denote by  $h' \in H^1(\mathbb{T}^s, \mathbb{Z})$  the homology class induced by  $\Gamma'$  on  $\mathbb{T}^s$ . As before an orbit enters along a normally hyperbolic weakly invariant manifold  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$ . As it diffused toward the center of a strong double resonance  $p_0$  the cylinder  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  becomes a flower cylinder and approaches to a small enough energy.

- If  $h$  and  $h'$  are simple, then we diffuse along a normally hyperbolic weakly invariant manifold  $\mathcal{M}_{h,\varepsilon}^{E_0,s}$  to the unique periodic orbit  $o_\varepsilon$ . This periodic orbit belongs to both  $\mathcal{M}_{h,\varepsilon}^{E_0,s}$  and  $\mathcal{M}_{h',\varepsilon}^{E_0,s}$  so we can “jump” from one cylinder to the other and continue diffusion along  $\mathcal{M}_{h',\varepsilon}^{E_0,s}$  (see Figure 12 in homologies and Figure 17, part c) in cohomologies).

- If  $h$  is non-simple and  $h'$  is simple and such that  $h' = h_1$  or  $h_2$  from the decomposition  $h = n_1 h_1 + n_2 h_2$ , then we can jump to  $\mathcal{M}_{h',\varepsilon}^{e,E_0}$  directly from  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  and cross the strong double resonance along  $\mathcal{M}_{h',\varepsilon}^{E_0}$ . (see Figure 12 in homologies and Figure 17, part b) in cohomologies).

- If  $h$  is non-simple and  $h'$  is simple, but neither  $h_1$  nor  $h_2$  from the decomposition  $h = n_1 h_1 + n_2 h_2$ , then we can jump to  $\mathcal{M}_{h_1,\varepsilon}^{e,E_0}$  from  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  and make a turn as in the first item (see Figure 12 in homologies and Figure 17, part d) in cohomologies).

- If both  $h$  and  $h'$  are non-simple, then both  $\mathcal{M}_{h,\varepsilon}^{e,E_0}$  and  $\mathcal{M}_{h',\varepsilon}^{e,E_0}$  becomes flower cylinders. In this case, there are two simple homology classes  $h_1$  and  $h'_1$  such that a normally hyperbolic (weakly) invariant manifolds  $\mathcal{M}_{h_1,\varepsilon}^{E_0,s}$  and  $\mathcal{M}_{h'_1,\varepsilon}^{E_0,s}$  crosses along the periodic orbit  $o_\varepsilon$ . Moreover, we can first jump onto  $\mathcal{M}_{h_1,\varepsilon}^{E_0,s}$  go to the periodic orbit along and jump to  $\mathcal{M}_{h'_1,\varepsilon}^{E_0,s}$ , cross (if necessary) the double resonance, and only afterward jump onto  $\mathcal{M}_{h',\varepsilon}^{e,E_0}$  (see Figure 12 in homologies and Figure 17, part e) in cohomologies).

## 4 Localization of the Aubry sets and the Mañe sets and Mather's projected graph theorems

We divide this section into two parts: single and double resonances.

In our proof we rely on various results about properties of the Aubry, Mather, and Mañe sets obtained earlier. This led to a notational conflict. Sometimes,  $\tilde{\mathcal{A}}(c)$ ,  $\tilde{\mathcal{M}}(c)$ ,  $\tilde{\mathcal{N}}(c)$  denote (continuous) Aubry, Mather, and Mañe sets as subsets of  $T^*\mathbb{T}^2 \times \mathbb{T} \supset \mathbb{T}^2 \times B^2 \times \mathbb{T}$  (see e.g. [57]). These are invariant sets of the associated Hamiltonian flow. Moreover, we need to keep track of time component (see sections 10 and 12). Sometimes,  $\tilde{\mathcal{A}}(c)$ ,  $\tilde{\mathcal{M}}(c)$ ,  $\tilde{\mathcal{N}}(c)$  denote (discrete) Aubry, Mather, and Mañe sets as subsets of  $T^*\mathbb{T}^2 \supset \mathbb{T}^2 \times B^2$  (see e.g. [9, 34]). These are invariant sets of the time one map. To somewhat consolidate both we denote

- $\tilde{\mathcal{A}}(c)$ ,  $\tilde{\mathcal{M}}(c)$ ,  $\tilde{\mathcal{N}}(c)$  the discrete Aubry, Mather, and Mañe sets respectively.
- $\tilde{\mathcal{A}}_H(c)$ ,  $\tilde{\mathcal{M}}_H(c)$ ,  $\tilde{\mathcal{N}}_H(c)$  the continuous Aubry, Mather, and Mañe sets respectively. Subscript  $H$  also emphasises dependence on the underlying Hamiltonian  $H$ .

### 4.1 Localization and Mather's projected graph theorem for single resonances

In this section we study the Hamiltonian  $H_\varepsilon$  near a single resonance  $\Gamma_j \subset \Gamma^*$  away from strong double resonances from the point of view of Mather theory and weak KAM theory. More precisely, we analyze dynamics with action component being in the neighborhood of the set

$$\{p = (p_*^s(p^f), p^f), p^f \in [a_-, a_+] \subset [a_{\min}, a_{\max}]\} \subset \Gamma_j.$$

The main goal here is to state Key Theorems 4 and 5. As before we assume that the averaged potential  $Z = Z_j$  satisfies the generic conditions [G0]-[G2]. Then that there exists a partition of  $[a_{\min}, a_{\max}] = \bigcup_{i=1}^{N-1} [a_i, a_{i+1}]$ , such that for  $p^f \in [a_i - \lambda, a_{i+1} + \lambda]$  the function  $Z(\theta^s, p_*^s(p^f), p^f)$  has a nondegenerate local maximum at  $\theta_i^s$  (see Figure 6).

Key Theorem 4 says that for one parameter family of  $c = (p_*^s(c^f), c^f)$  and  $c^f \in [a_i - \lambda, a_{i+1} + \lambda]$  we have that the corresponding Aubry sets  $\tilde{\mathcal{A}}_H(c)$  belong to a neighborhood of the normally hyperbolic invariant cylinders  $\mathcal{C}_i^{(j)}$  constructed in Key Theorem 1. Then Key Theorem 5 improves this claim to say that these  $\tilde{\mathcal{A}}_H(c)$ 's and even “bigger” Mañe sets  $\tilde{\mathcal{N}}_H(c)$  belong to the corresponding cylinder  $\mathcal{C}_i^{(j)}$  away from bifurcation points  $a_i, a_{i+1}$ . Near a bifurcation point  $a_{i+1}$  they could belong to either of the two cylinders  $\mathcal{C}_i^{(j)}$  and  $\mathcal{C}_{i+1}^{(j)}$ . Moreover, in all cases these Aubry sets

$\tilde{\mathcal{A}}_H(c) \cap \mathcal{C}_i^{(j)}$  has a one-to-one projection onto the fast angle  $\theta^f$  with Lipschitz inverse. In a general setting this is the well-known Mather graph theorem. In our setting we call it *Mather's projected graph theorem*. It leads to ordering of minimizers on the base  $(\theta^f, t) \in \mathbb{T}^2$  and shows that such Aubry sets  $\tilde{\mathcal{A}}_H(c) \cap \mathcal{C}_i^{(j)}$  can be only of Aubry-Mather type: either invariant 2-dimensional tori, or Denjoy sets, or periodic orbits with connecting heteroclinics.

It turns out that under some circumstances it is more convenient to study continuous dynamics in  $\mathbb{T}^2 \times B^2 \times \mathbb{T} \subset T^*\mathbb{T}^2 \times \mathbb{T}$ . In other situations one considers the time one map and study discrete dynamics in  $\mathbb{T}^2 \times B^2 \rightarrow \mathbb{T}^2 \times B^2$ . As a result there is a one-to-one correspondence between invariant sets of the continuous flow and its discrete version.

We first point out the following consequences of the genericity conditions [G0]-[G2]: there exists  $0 < b < \delta/2 < \lambda/4$  depending on  $H_1$  such that

[G1']

$$Z(\theta_i^s(p^f), p_*^s(p^f)) - Z(\theta^s, p_*^s(p^f)) \geq b \|\theta^s - \theta_i^s(p^f)\|,$$

for each  $p^f \in [a_i + b, a_{i+1} - b]$ .

[G2'] For  $p^f \in [a_{i+1} - b, a_{i+1} + b]$ ,  $i = 0, \dots, s-2$ , we have

$$\begin{aligned} \max\{Z(\theta_i^s, p_*^s(p^f)), Z(\theta_{i+1}^s, p_*^s(p^f))\} - Z(\theta^s, p_*^s(p^f)) \\ \geq b \min\{\|\theta^s - \theta_i^s\|, \|\theta^s - \theta_{i+1}^s\|\}^2. \end{aligned}$$

In the first case, the function  $Z$  has a single non-degenerate maximum, which we will call *the "single peak" case*, while the second case will be called *the "double peak" case*. The single peak case corresponds to a unique maximum for  $Z$ , and the double peak case corresponds to a neighborhood of a bifurcation. The shape of the function  $Z$  allows us to localize properly chosen Aubry sets and Mañe sets of the Hamiltonian  $H_\epsilon$ .

Recall that  $[a_-^i, a_+^i]$  denotes either a non-boundary segment  $[a_i^{(j)} - \delta, a_{i+1}^{(j)} + \delta]$ ,  $i \in \{1, \dots, M-1\}$  or one of boundary segments  $[a_0^{(j)} - 3\sqrt{\epsilon}, a_1^{(j)} + \delta]$  and  $[a_{M-1}^{(j)} - \delta, a_M^{(j)} + 3\sqrt{\epsilon}]$ . According to Key Theorem 1, for each  $[a_-^i, a_+^i]$  there exists

$$\mathcal{C}_i^{(j)} = \{(\theta^s, p^s) = (\Theta_j^s, P_j^s)(\theta^f, p^f, t) : p^f \in [a_-^i, a_+^i], (\theta^f, t) \in \mathbb{T} \times \mathbb{T}\},$$

which contains all full orbits contained in

$$V_i^{(j)} := \{(\theta, p, t) : p^f \in [a_-^i, a_+^i], \|(\theta^s, p^s) - (\theta_j^s, p_j^s)\| \leq \rho_1\}.$$

As there may be orbits in  $\mathcal{C}_i^{(j)}$  escaping from  $V_i^{(j)}$  through top/bottom boundaries  $p^f = a_-^i$  or  $a_+^i$ . Therefore,  $\mathcal{C}_i^{(j)}$  is not necessarily invariant in the strict sense.

This information allows us to study the Mather set, Aubry set and Mañe set of the Hamiltonian  $H_\epsilon$ . Let  $A = 2 \max_{p \in B^2, v \in \mathbb{R}^2, |v|=1} \langle \partial_{pp}^2 H_0(p)v, v \rangle$ .

Let  $b > 0$ . Denote  $[\bar{a}_-, \bar{a}_+^i]$  denotes either a non-boundary segment  $[a_i^{(j)} - b, a_{i+1}^{(j)} + b]$ ,  $i \in \{1, \dots, M-1\}$  or one of boundary segments  $[a_0^{(j)} - 2\sqrt{\epsilon}, a_1^{(j)} + b]$  and  $[a_{M-1}^{(j)} - b, a_M^{(j)} + \sqrt{\epsilon}]$ .

**Key Theorem 4** (Localization). *Let the averaged potential  $Z$  of  $H_\epsilon$ , given by (1), satisfy  $[G0]$ - $[G2]$ , then there exist  $\delta_0 = \delta_0(H_0, \lambda, r, \Gamma^*) > 0$ ,  $\epsilon_0 = \epsilon_0(H_0, \lambda, r, \Gamma^*) > 0$ , and  $0 < \rho_2 < \rho_1$  such that for  $0 < \epsilon < \epsilon_0$  and  $0 < b < \delta/2 < \delta_0/2$  for the Hamiltonian  $H_\epsilon$  the following hold.*

1. For any  $c = (p_*^s(c^f), c^f)$  such that  $c^f \in [\bar{a}_-, \bar{a}_+^i]$ ,  $\tilde{\mathcal{N}}_H(c)$  is contained in

$$\{(\theta, p, t) : \|p - c\| \leq (10A + 1)\sqrt{\epsilon}, \|\theta^s - \theta_j^s(p^f)\| \leq \rho_2\}.$$

2. For  $c = (p_*^s(c^f), c^f)$  such that  $c^f \in [\bar{a}_-, \bar{a}_+^i]$ , we have that  $\tilde{\mathcal{A}}_H(c)$  is contained in

$$\{(\theta, p, t) : \|p - c\| \leq (10A + 1)\sqrt{\epsilon}, \min\{\|\theta^s - \theta_i^s(p^f)\|, \|\theta^s - \theta_{i+1}^s(p^f)\|\} \leq \rho_2\}.$$

Apply the statements of the previous theorem with Key Theorem 1, we may further localize these sets on the normally hyperbolic weakly invariant cylinders  $\mathcal{C}_i^{(j)}$  and  $\mathcal{C}_i^{(j+1)}$ . Moreover, locally these sets are graphs over the  $\theta^f$  component, which is a version of Mather's projected graph theorem.

Let  $\delta' > 0$ . Denote  $[\hat{a}_-, \hat{a}_+^i]$  denotes either a non-boundary segment  $[a_i^{(j)} - \delta', a_{i+1}^{(j)} + \delta']$ ,  $i \in \{1, \dots, M-1\}$  or one of boundary segments  $[a_0^{(j)} - \sqrt{\epsilon}, a_1^{(j)} + \delta']$  and  $[a_{M-1}^{(j)} - \delta', a_M^{(j)} + \sqrt{\epsilon}]$ .

**Key Theorem 5** (Mather's projected graph theorem). *Let the averaged potential  $Z$  of  $H_\epsilon$ , given by (1), satisfy  $[G0]$ - $[G2]$ , there exist  $\delta' = \delta'(b, H_0, \lambda, r, \Gamma^*) > 0$  and  $\epsilon_0 = \epsilon_0(b, H_0, \lambda, r, \Gamma^*) > 0$  such that for  $0 < \delta \leq \delta_0$  and  $0 < \epsilon < \epsilon_0$  we have:*

1. There exists  $0 < \rho_2 < \rho_1$  such that for  $c = (p_*^s(c^f), c^f)$  with  $c^f \in [\hat{a}_-, \hat{a}_+^i]$  the Mañe set  $\tilde{\mathcal{N}}_H(c) \supset \tilde{\mathcal{A}}_H(c)$  is contained in the normally hyperbolic weakly invariant cylinder  $\mathcal{C}_i^{(j)}$ <sup>12</sup>.

Moreover, let  $\pi_{\theta^f}$  be the projection to the  $\theta^f$  component, we have that  $\pi_{\theta^f}|_{\tilde{\mathcal{A}}_H(c)}$  is one-to-one and the inverse is Lipschitz.

2. For  $c^f \in [\hat{a}_+^i - \delta', \hat{a}_+^i + \delta']$ , we have that  $\tilde{\mathcal{A}}_H(c) \subset \mathcal{C}_i^{(j)} \cup \mathcal{C}_{i+1}^{(j)}$ .

Moreover,  $\pi_{\theta^f}|_{\tilde{\mathcal{A}}_H(c) \cap \mathcal{C}_i^{(j)}}$  and  $\pi_{\theta^f}|_{\tilde{\mathcal{A}}_H(c) \cap \mathcal{C}_{i+1}^{(j)}}$  are both one-to-one and have Lipschitz inverses.

---

<sup>12</sup>as before “weakly” means that the Hamiltonian vector field of  $H_\epsilon$  is tangent to  $\mathcal{C}_i^{(j)}$ .

Recall that Key Theorem 1 for resonant segments not ending at a strong double resonance follows from Theorem 4.1 [13]. For resonant segments ending at a strong double resonance it follows from Theorem 15. Proofs of both Theorem 4.1 [13] and Theorem 15 consists of two steps: find a normal form  $N_\varepsilon = H_\varepsilon \circ \Phi_\varepsilon$  (Corollary 3.2 [13] and Theorem 17 respectively) and construct an isolating block for  $N_\varepsilon$  (see sections 4 in [13] and section 7.3 respectively). The only difference of the two is that in the normal form theorem we show that

$$N_\varepsilon = H_\varepsilon \circ \Phi_\varepsilon = H_0(\cdot) + \varepsilon Z(\cdot) + \varepsilon R(\cdot, t),$$

where  $\|R\| \leq \delta$  for some small predetermined  $\delta$  and two different norms. In [13] we use the standard  $C^2$ -norm, while in section 7.3 we use a skew-symmetric  $C^2$ -norm. It turns out it does not affect applicability of the isolating block arguments as shown in section 7.3.

We observe that Key Theorem 4 can be proven in exactly the same way as Theorem 5.1 [13] and Key Theorem 5 can be proven in exactly the same way as Theorem 5.2 [13]. To see that notice that Theorems 5.1 and 5.2 are proven in section 5 and the proof applies to the remainder  $R$  being small in the  $C^0$ -norm (see Lemma 5.3 and Proposition 5.6 there). It remains to note that the skew-symmetric norm from section 7.3 coincides with the  $C^0$ -norm from section 4 in [13]. Thus, making the proof from section 4 [13] applicable to our situation.

## 4.2 Localization and Mather's projected graph theorem for double resonances

In this section we study the Hamiltonian  $H_\varepsilon$  near a strong double resonance from the point of view of Mather and weak KAM theory. More precisely, we consider a double resonance  $p_0 \in \Gamma \cap \Gamma'$  and dynamics in its  $O(\sqrt{\varepsilon})$ -neighborhood. As we pointed out in (5) and remark 3.5 this dynamics is closely related to dynamics of the slow mechanical system  $H^s(I^s, \varphi^s)$ . Thus, we consider the following Hamiltonian

$$H_\varepsilon^s(\varphi^s, I^s, \tau) = K(I^s) - U(\varphi^s) + \sqrt{\varepsilon}P(\varphi^s, I^s, \tau), \quad (7)$$

where  $\varphi^s \in \mathbb{T}^s$ ,  $I^s \in \mathbb{R}^2$  and  $\tau \in \sqrt{\varepsilon}\mathbb{T}$ , with  $\|P\|_{C^2} \leq 1$ . Denote  $H^s(\varphi^s, I^s) = K(I^s) - U(\varphi^s)$ . Without loss of generality we assume  $\min U = 0$ . As before we assume that minimum is unique.

By the Maupertuis principle, for each  $E > 0$  and  $h \in H_1(\mathbb{T}^s, \mathbb{Z})$ , there exists at least one minimal closed geodesic in the homology class  $h$ . For the geodesic on the critical energy surface there are three cases:

- simple loop passing through the origin,



- simple loop not passing the origin,
- non-simple loop.

We first discuss the simple loop case.

#### 4.2.1 NHIMs near a double resonance

The net of NHIMs near a double resonance consists of high energy and low energy cylinders, and for low energy, the type of the cylinders depends on the properties of the associated homologies  $h$  and  $h'$ . We summarize the content of Key Theorems 2 and 3 as follows.

For the slow system, there exists NHICs

$$\mathcal{M}_h^{E_j, E_{j+1}} = \bigcup_{E_j - \delta < E < E_{j+1} + \delta} \gamma_h^E, \quad 0 \leq j \leq N-1$$

and the corresponding cylinders  $\mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}$  for the perturbed system (constructed in Key Theorem 2). Let  $\mathcal{I} : (\varphi^s, I^s) \mapsto (\varphi^s, -I^s)$ , then by the symmetry of the slow system  $H^s$ , the cylinders  $\mathcal{M}_{-h}^{E_j, E_{j+1}} = \mathcal{I}(\mathcal{M}_h^{E_j, E_{j+1}})$ . The perturbed cylinders  $\mathcal{M}_{-h,\epsilon}^{E_j, E_{j+1}}$  are graphs over  $\mathcal{M}_{-h}^{E_j, E_{j+1}}$ , but is not the reflection of  $\mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}$  in general. In addition, we assume that  $E_0$  is not an bifurcation value. Otherwise, we relabel  $E_0$  as  $E_1$  and pick a smaller  $E_0$ .

In the case that  $h$  is simple and non-critical, the cylinder  $\mathcal{M}_h^{E_0, E_1}$  is smoothly attached to the cylinder

$$\mathcal{M}_h^{0, E_0} = \bigcup_{-\delta < E < E_0 + \delta} \gamma_h^E.$$

If  $h$  is simple and critical, by Key Theorem 3, there exists a simple loop cylinder  $\mathcal{M}_h^{E_0, s}$  containing the loop  $\gamma_h^0$ . The cylinder is smoothly attached to both  $\mathcal{M}_h^{E_0, E_1}$  and  $\mathcal{M}_{-h}^{E_0, E_1}$ .

If  $h$  is non-simple, by Key Theorem 3, part 2, there exists an invariant cylinder

$$\mathcal{M}_h^{e/2, E_0} = \bigcup_{e/2 < E < E_0} \gamma_h^E.$$

By local uniqueness,  $\mathcal{M}_h^{e/2, E_0}$  is smoothly attached to  $\mathcal{M}_h^{E_0, E_1}$ . For a sufficiently small  $\epsilon$ , the perturbed cylinder  $\mathcal{M}_{h,\epsilon}^{e, E_0}$  is well defined.

After the perturbation, the hyperbolic fixed point  $(0, 0)$  corresponds to a hyperbolic periodic orbit, denoted  $o_\epsilon$ . If  $h$  is simple and critical,  $o_\epsilon \subset \mathcal{M}_{h,\epsilon}^{E_0, s}$ ; but if  $h$  is simple and non-critical,  $o_\epsilon \cap \mathcal{M}_{h,\epsilon}^{0, E_1} = \emptyset$ .

For simple loops, we have a net of NHIMs

$$\{\mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}\}_{j=0}^N \cup \mathcal{M}_{h,\epsilon}^{E_0, s} \cup \{\mathcal{M}_{-h,\epsilon}^{E_j, E_{j+1}}\}_{j=0}^N$$

that connects the high energies of homology  $h$  and homology  $-h$ . For non-simple loop, the NHIMs does not reach critical energy. However, as  $h$  decomposes to  $n_1 h_1 + n_2 h_2$ , with  $h_1$  and  $h_2$  simple, we have the corresponding simple cylinders  $\mathcal{M}_{h_1,\epsilon}^{E_0, s}$  and  $\mathcal{M}_{h_2,\epsilon}^{E_0, s}$ . For all types of cylinders, we have localization of the Aubry sets and the Mañe sets, with appropriately chosen cohomologies.

#### 4.2.2 Choice of the cohomology classes

For the slow system  $H^s$ , each minimal geodesic  $\gamma_h^E$  corresponds to a minimal measure of the system, and has an associated cohomology class. More precisely, we assume that  $\gamma_h^E$  is parametrized so that it satisfies the Euler-Lagrange equation, and  $T = T(\gamma_h^E)$  is its period under this parametrization. Then the probability measure supported on  $\gamma_h^E$  is a minimal measure, and its rotation number is

$$\frac{1}{T} \int_0^T \dot{\gamma}_h^E dt = \frac{1}{T}.$$

The associated cohomology class is a convex subset

$$\mathcal{LF}_\beta(h/T(\gamma_h^E)),$$

of  $H^1(\mathbb{T}^2, \mathbb{R})$ , where  $\mathcal{LF}_\beta$  is the Legendre-Fenchel transform of the  $\beta$ -function defined by Mather (see (30) section 9.6 for definition).

Assume that the system  $H^s$  satisfies conditions [DR1]-[DR3] and conditions [A0]-[A4]. Then for  $0 < E \leq E_0$ , or  $E_j < E < E_{j+1}$ ,  $0 \leq j \leq N-1$ , there exists a unique minimal geodesic  $\gamma_h^E$  for energy  $E$ . In this case, we define

$$\lambda_h^E = 1/T(\gamma_h^E).$$

For the bifurcation value  $E = E_j$ ,  $0 \leq j \leq N-1$ , there are two minimal geodesics  $\gamma_h^E$  and  $\bar{\gamma}_h^E$ . We still write  $\lambda_h^E = 1/T(\gamma_h^E)$ , where the choice of  $\gamma_h^E$  among the two is arbitrary. We will show that the set  $\mathcal{LF}_\beta(\lambda_h^E h)$  is independent of the choice of  $\gamma_h^E$  (see Theorem 28 section C), and hence  $\mathcal{LF}_\beta(\lambda_h^E h)$  is well defined as a set function of  $E$ .

We call the union

$$\bigcup_{E>0} \mathcal{LF}_\beta(\lambda_h^E h) \tag{8}$$

the *channel* associated to the homology  $h$ , and we will choose a curve of cohomologies within this channel. For our choice of cohomologies, the associated Aubry sets are contained in the normally hyperbolic invariant cylinders. The choice of the cohomology in the channel is illustrated in Figure 13.

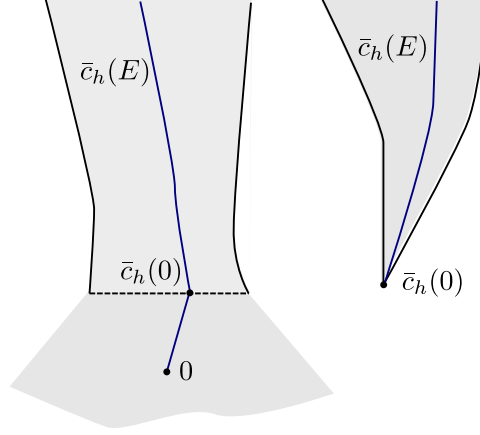


Figure 13: Channel and cohomology: *Left: simple channel (both critical and non-critical); right: non-simple channel*

**Proposition 4.1.** *Assume that  $H^s$  satisfies the conditions [DR1]-[DR3] and [A0]-[A4]. Then there exists a continuous function  $\bar{c}_h : [0, \bar{E}] \rightarrow H^1(\mathbb{T}^s, \mathbb{R})$  satisfying  $\bar{c}_h(E) \in \mathcal{LF}_\beta(\lambda_h^E h)$ ,  $E > 0$ , with the following properties.*

1. *For  $0 < E \leq E_0$ , or  $E_j < E < E_{j+1}$ ,  $0 \leq j \leq N - 1$ ,*

$$\mathcal{A}_{H^s}(\bar{c}_h(E)) = \gamma_h^E.$$

2. *For the bifurcation values  $E = E_j$ ,  $j \geq 1$ ,*

$$\mathcal{A}_{H^s}(\bar{c}_h(E_j)) = \gamma_h^E \cup \bar{\gamma}_h^E.$$

3. *If  $h$  is simple and critical, then*

$$\mathcal{A}_{H^s}(\bar{c}_h(0)) = \gamma_h^0;$$

*for each  $0 \leq \lambda < 1$ ,*

$$\mathcal{A}_{H^s}(\lambda \bar{c}_h(0)) = \{\varphi^s = 0\}.$$

4. *If  $h$  is simple and non-critical, then*

$$\mathcal{A}_{H^s}(\bar{c}_h(0)) = \gamma_h^0 \cup \{\varphi^s = 0\};$$

*for each  $0 \leq \lambda < 1$ ,*

$$\mathcal{A}_{H^s}(\lambda \bar{c}_h(0)) = \{\varphi^s = 0\}.$$

### 4.2.3 Properties of the Aubry sets and Mañe sets

For the perturbed slow system  $H_\epsilon^s = H_s + \sqrt{\epsilon}P$  on  $\mathbb{T}^s \times \mathbb{R}^2 \times \sqrt{\epsilon}\mathbb{T}$ , similar to Proposition 4.1, we have the following localization statement.

**Theorem 8.** *Assume that  $H^s$  satisfies the conditions of Key Theorems 2 and 3. Then there exists a constant  $\delta = \delta(H^s) > 0$  such that for some  $\varepsilon_0 = \varepsilon_0(H^s, \delta) > 0$  and for all  $0 < \epsilon < \epsilon_0$  we have*

1. For  $E_j + \delta \leq E \leq E_{j+1} - \delta$  with  $0 \leq j \leq N$ ,

$$\tilde{\mathcal{N}}_{H_\epsilon^s}(\bar{c}_h(E)) \subset \mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}.$$

2. For  $E_j - \delta < E < E_j + \delta$  with  $1 \leq j \leq N$ ,

$$\tilde{\mathcal{A}}_{H_\epsilon^s}(\bar{c}_h(E)) \subset \mathcal{M}_{h,\epsilon}^{E_{j-1}, E_j} \cup \mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}.$$

3. Assume that  $h$  is simple and critical, then

$$\tilde{\mathcal{A}}_{H_\epsilon^s}(\bar{c}_h(E)) \subset \mathcal{M}_{h,\epsilon}^{E_0, s} \cup \mathcal{M}_{h,\epsilon}^{E_0, E_1}, \quad 0 \leq E \leq E_0,$$

$$\tilde{\mathcal{A}}_{H_\epsilon^s}(\lambda \bar{c}_h(0)) \subset \mathcal{M}_{h,\epsilon}^{E_0, s} \cup \mathcal{M}_{h,\epsilon}^{E_0, E_1}, \quad 0 \leq \lambda < 1.$$

Note that  $\mathcal{M}_{h,\epsilon}^{E_0, s} \cup \mathcal{M}_{h,\epsilon}^{E_0, E_1}$  is one smooth cylinder.

4. If  $h$  is simple and non-critical, then

$$\tilde{\mathcal{N}}_{H_\epsilon^s}(\bar{c}_h(E)) \subset \mathcal{M}_{h,\epsilon}^{0, E_0}, \quad \delta \leq E \leq E_0,$$

$$\tilde{\mathcal{A}}_{H_\epsilon^s}(\bar{c}_h(E)) \subset o_\epsilon \cup \mathcal{M}_{h,\epsilon}^{0, E_0}, \quad 0 \leq E < \delta,$$

$$\tilde{\mathcal{A}}_{H_\epsilon^s}(\lambda \bar{c}_h(0)) \subset o_\epsilon \cup \mathcal{M}_{h,\epsilon}^{0, E_0}, \quad 1 - \delta \leq \lambda < 1,$$

$$\tilde{\mathcal{N}}_{H_\epsilon^s}(\lambda \bar{c}_h(0)) \subset o_\epsilon, \quad 0 \leq \lambda < 1 - \delta.$$

5. If  $h$  is non-simple, then

$$\tilde{\mathcal{N}}_{H_\epsilon^s}(\bar{c}_h(E)) \subset \mathcal{M}_{h,\epsilon}^{e, E_0}, \quad e \leq E \leq E_0.$$

As explained in Appendix B.3, there is a relation between the Aubry sets of the Hamiltonian  $H_\epsilon^s$  and the Hamiltonian  $H_\epsilon$  in  $(\varphi^s, p^s, t)$  coordinates. More precisely, we have

$$\mathcal{A}_{H_\epsilon}(c_h(E)) = \Phi_L^{-1} \mathcal{A}_{H_\epsilon^s}(\bar{c}_h(E)), \quad \text{with } c_h(E) = p_0 + \bar{c}_h(E)B^T\sqrt{\epsilon}, \quad (9)$$

where  $\Phi_L(\varphi^s, p^s) = (\varphi^s, (p^s - p_0^s)/\sqrt{\epsilon})$ . Denote  $c_h^\lambda = p_0 + \lambda B^T \bar{c}_h(0)\sqrt{\epsilon}$ , we have  $\mathcal{A}_{H_\epsilon}(c_h^\lambda) = \Phi_L^{-1} \mathcal{A}_{H_\epsilon^s}(\lambda \bar{c}_h(0))$  as well. Similar conclusions hold for the Mañe sets.

As a consequence, we obtain localization statements about cohomology classes of the original Hamiltonian  $H_\epsilon$ .

**Key Theorem 6.** *Assume that  $H^s$  satisfies the conditions of Key Theorems 2 and 3. Then there exists  $\delta = \delta(H_0, H_1, \Gamma^*, r) > 0$  such that for some  $\varepsilon_0 = \varepsilon_0(H_0, H_1, \Gamma^*, r, \delta) > 0$  and for all  $0 < \epsilon < \epsilon_0$  we have*

1. *For  $E_j + \delta \leq E \leq E_{j+1} - \delta$  with  $0 \leq j \leq N$  we have*

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h(E)) \subset \mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}.$$

2. *For  $E_j - \delta < E < E_j + \delta$  with  $1 \leq j \leq N$ ,*

$$\tilde{\mathcal{A}}_{H_\epsilon}(c_h(E)) \subset \mathcal{M}_{h,\epsilon}^{E_{j-1}, E_j} \cup \mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}.$$

3. *Assume that  $h$  is simple and critical, then*

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h(E)) \subset \mathcal{M}_{h,\epsilon}^{E_0, s} \cup \mathcal{M}_{h,\epsilon}^{E_0, E_1}, \quad 0 \leq E \leq E_0,$$

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h^\lambda) \subset \mathcal{M}_{h,\epsilon}^{E_0, s}, \quad 0 \leq \lambda < 1.$$

4. *If  $h$  is simple and non-critical, then*

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h(E)) \subset \mathcal{M}_{h,\epsilon}^{0, E_0}, \quad \delta \leq E \leq E_0,$$

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h(E)) \subset o_\epsilon \cup \mathcal{M}_{h,\epsilon}^{0, E_0}, \quad 0 \leq E < \delta,$$

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h^\lambda) \subset o_\epsilon \cup \mathcal{M}_{h,\epsilon}^{0, E_0}, \quad 1 - \delta \leq \lambda < 1,$$

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h^\lambda) \subset o_\epsilon, \quad 0 \leq \lambda < 1 - \delta.$$

5. *If  $h$  is non-simple, then*

$$\tilde{\mathcal{N}}_{H_\epsilon}(c_h(E)) \subset \mathcal{M}_{h,\epsilon}^{e, E_0}, \quad e \leq E \leq E_0.$$

We have that the Aubry sets satisfy Mather's projected graph theorem.

**Key Theorem 7** (Mather's projected graph theorem). *Assume that  $H^s$  satisfies the conditions of Key Theorem 2 and 3. Then there exist  $\delta = \delta(H_0, H_1, \Gamma, r) > 0$  and  $\epsilon_0 = \epsilon_0(H_0, H_1, \Gamma, r, \delta) > 0$  such that for all  $0 < \epsilon < \epsilon_0$ ,*

1. *For  $E_j + \delta \leq E \leq E_{j+1} - \delta$  with  $0 \leq j \leq N$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h(E))$  is contained in a Lipschitz graph over  $\gamma_h^E$ .*
2. *For  $E_j - \delta < E < E_j + \delta$  with  $1 \leq j \leq N$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h(E)) \cap \mathcal{M}_{h,\epsilon}^{E_{j-1}, E_j}$  and  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h(E)) \cap \mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}$  are contained in graphs over  $\gamma_h^E$  and  $\bar{\gamma}_h^E$ , respectively.*

3. Assume that  $h$  is simple and critical, then

- for  $0 \leq E \leq E_0$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h(E))$  is contained in a graph over  $\gamma_h^E$ ;
- for  $0 \leq \lambda < 1$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h^\lambda)$  is contained in a graph over  $\gamma_h^0$ .

4. If  $h$  is simple and non-critical, then

- for  $0 \leq E \leq E_0$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h(E))$  is contained in a graph over  $\gamma_h^E$ ;
- for  $1 - \delta \leq \lambda < 1$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h^\lambda) \cap \mathcal{M}_{h,\epsilon}^{0,E_0}$  is contained in a graph over  $\gamma_h^0$ ;
- for  $0 \leq \lambda \leq 1 - \delta$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h^\lambda) \subset o_\epsilon$ .

5. If  $h$  is non-simple, then for  $e \leq E \leq E_0$ , the Aubry set  $\tilde{\mathcal{A}}_{H_\epsilon}(c_h(E))$  is contained in a graph over  $\gamma_h^E$ .

### 4.3 Choice of auxiliary cohomology classes for a non-simple homology

Assume that  $h$  is a non-simple homology. Assume that condition [A0] holds and by Lemma 3.2 we have decomposition  $h = n_1 h_1 + n_2 h_2$  into simple homologies. Let  $\bar{c}_h(E)$  be the associated cohomologies as in Proposition 4.1. Since the homology  $h_1$  is simple, we can choose a curve of cohomology  $c_{h_1}(E)$  contained in the channel associated to  $h_1$  satisfying the conclusions of Proposition 4.1. In this section, we show how to modify the function  $\bar{c}_{h_1}(E)$  such that it satisfies some additional properties relative to the homology  $h$ .

Let  $h_1^\perp \in H^1(\mathbb{T}^2, \mathbb{R})$  be a unit homology vector orthogonal to  $h_1$ .

**Proposition 4.2.** *There exists a continuous function  $\bar{b}_{h_1} : [0, E_0] \rightarrow H^1(\mathbb{T}^2, \mathbb{R})$ ,  $\bar{b}_{h_1}(E) \in \mathcal{LF}_\beta(\lambda_{h_1}^E h_1)$  for  $E > 0$ , with the following properties.*

1. For  $0 < E \leq E_0$ ,  $\mathcal{A}_{H^s}(\bar{b}_{h_1}(E)) = \gamma_{h_1}^E$ .
2.  $\bar{b}_{h_1}(0) = \bar{c}_h(0)$ .
- 3.

$$\lim_{E \rightarrow 0+} \frac{\bar{b}_{h_1}(E) - \bar{c}_h(E)}{\|\bar{b}_{h_1}(E) - \bar{c}_h(E)\|} = h_1^\perp.$$

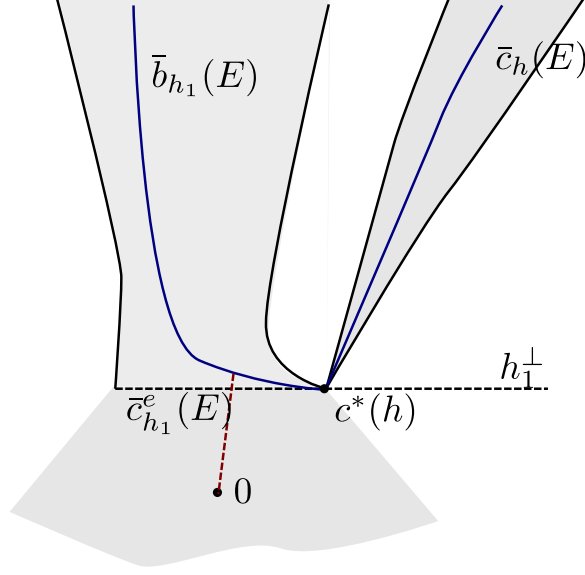


Figure 14: Simple channel associated to a non-simple one

In Appendix C we analyze properties of the channel  $\bigcup_{E>0} \mathcal{LF}_\beta(\lambda_h^E h')$  of cohomologies, defined in (8), for simple and non-simple  $h$ . Most of them are summarized in Theorem 28. Using this Theorem we can choose  $\bar{b}_{h_1}(E)$  in such a way that the following properties of the channels:

- The set  $\mathcal{LF}_\beta(\lambda_h^E h)$  is a closed interval of non-zero width for each  $E > 0$ .
- As  $E \rightarrow 0$ , the set  $\mathcal{LF}_\beta(\lambda_h^E h)$  converges to a single point  $c^*(h)$ .
- The set  $\mathcal{LF}_\beta(\lambda_{h_1}^E h_1)$  is a closed interval of non-zero width. Hence, the channel  $\bigcup_{E>0} \mathcal{LF}_\beta(\lambda_{h_1}^E h_1)$  has width bounded away from zero.
- The segment  $\mathcal{LF}_\beta(\lambda_{h_1}^E h_1)$  is parallel to  $h_1^\perp$ , where  $h_1^\perp$  denotes the vector perpendicular to  $h_1$ . As  $E \rightarrow 0$ , the set  $\mathcal{LF}_\beta(\lambda_{h_1}^E h_1)$  converges to a segment of nonzero width. Moreover, one of the end points of this segment is  $c_h^*(h)$ .

For an illustration of the channels, see Figure 14.

It is shown in section 3.3 that under our non-degeneracy conditions,  $\gamma_{h_1}^0$  and  $\gamma_{h_2}^0$  are both tangent to a common direction, let's denote it  $v_0$ . We will assume the following.

*The vector  $v_0$  is not parallel to  $h_1$ .*

However, this assumption is not restrictive, if this condition is not satisfied by  $h_1$ , we will simply switch the names of  $h_1$  and  $h_2$ .

We remark that the function  $\bar{b}_{h_1}$  does not satisfy the conclusions of the Proposition 4.1, as the curve approaches the boundary of the channel, instead of staying in the interior. Indeed, since  $\bar{b}_{h_1}(0) = \bar{c}_h(0)$ , then  $\mathcal{A}_{H^s}(\bar{b}_{h_1}(E)) = \gamma_{h_1}^E \cup \gamma_{h_2}^E$ , instead of being  $\gamma_{h_1}^E$ . However, we can modify the function  $\bar{b}_{h_1}$  near  $E = 0$  such that it satisfies the conclusions of the Proposition 4.1.

**Proposition 4.3.** *For any  $e > 0$ , there exists a function  $\bar{c}_{h_1} : [0, E_0] \longrightarrow H^1(\mathbb{T}^2, \mathbb{R})$ ,  $\bar{c}_{h_1}^e(E) \in \mathcal{LF}_\beta(\lambda_{h_1}^E h_1)$  for  $E > 0$ , such that*

$$\bar{c}_{h_1}^e(E) = \bar{b}_{h_1}(E), \quad e \leq E \leq E_0,$$

*and  $\bar{c}_{h_1}^e(E)$  satisfies the conclusions of Proposition 4.1.*

The modification is illustrated in Figure 14. For the purpose of diffusion, we will “jump” from the cohomology  $\bar{c}_h(E)$  to  $\bar{b}_{h_1}(E)$  at some energy  $e_0 > e$ . We then follow the modified cohomology curve  $\bar{c}_{h_1}^e(E)$  towards  $c = 0$ .

We define the cohomology  $c_{h_1}^e(E)$  and  $c_{h_1}^{e,\lambda}$  for the original coordinates in the same way as in section 4.2.3.

**Remark 4.1.** *As the proof of Key Theorems 7 and 6 depend only on the conclusions of Proposition 4.1, for our choice of cohomology  $c_{h_1}$ , the conclusions of these Key Theorems also hold.*



## 5 Description of $c$ -equivalence and a variational $\lambda$ -lemma

### 5.1 Heuristic descriptions

We start this section with an heuristic description of  $c$ -equivalence as digesting definitions and abstract objects involved here is a nontrivial task. To motivate a variational  $\lambda$ -lemma we start with a simple case the standard  $\lambda$ -lemma.

Consider a smooth twist map  $f : \mathcal{A} \longrightarrow \mathcal{A}$ ,  $\mathcal{A} = \mathbb{T} \times \mathbb{R} \ni (\theta, I)$  satisfying the standard assumptions of the Aubry-Mather theory. Suppose  $\{x_i\}_{i=1}^s$  is a collection of periodic points so that each  $x_i$

- is minimal,
- has rotation number  $\omega_i = p_i/q_i$ , and
- non-degenerate, i.e. it is a saddle and has smooth local stable and unstable manifolds  $W_{loc}^s(x_i)$  and  $W_{loc}^u(x_i)$  resp.

Due to minimality we know that  $W_{loc}^{s/u}(x_i)$  are smooth graphs over  $\mathbb{T}$ . Suppose also that

$$f^{k_i}(W_{loc}^u(x_i)) \quad \text{and} \quad W_{loc}^s(x_{i+1}) \quad \text{intersect transversally}$$

for each  $i = 1, \dots, s-1$ . Then we have the following: for some  $N$  there is a local graph

$$\overline{\mathcal{G}}' \subset \cup_{j=1}^N f^j(W_{loc}^u(x_1))$$

such that it is  $C^1$ -close to  $W_{loc}^u(x_s)$ . Moreover, this is stable property, i.e. we can choose a local graph  $\mathcal{G}$  which is  $C^1$ -close to  $W_{loc}^u(x_1)$  and it satisfies the same property. It turns out this can be included into a general framework of weak KAM theory of Fathi [34].

When one considers a unstable manifold  $W^u(x_i)$  of a minimal periodic orbits, usually it is not a graph. However, there is a part of it, which is a graph with discontinuities (see Figure 15). We also have that locally (in some open set  $U$ ) unstable manifold can be given

$$W_{loc}^u = \{(x, c + du_x) : x \in U\},$$

where  $du_x$  is the gradient of  $u$  at  $x$ .

This motivates a definition of an *overlapping pseudograph*. To put things in a general framework let  $M$  be a compact manifold,  $TM$  is a tangent bundle to  $M$ , and

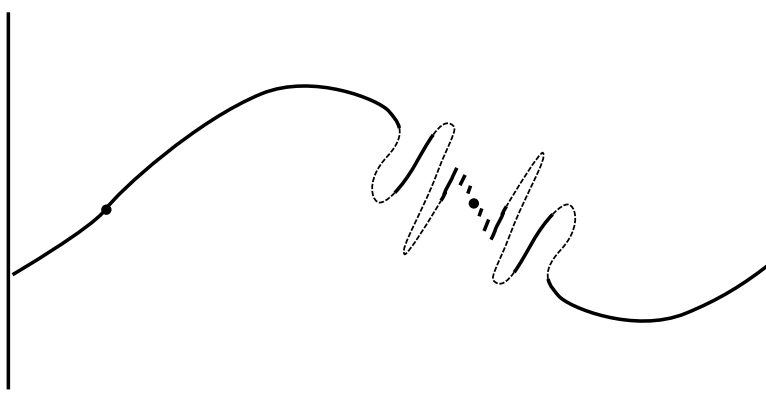


Figure 15: Pseudograph for a fixed point of a twist map

$\pi : TM \longrightarrow M$  is the natural projection. Given a Lipschitz function  $u : M \longrightarrow \mathbb{R}$  and a closed smooth form  $\eta$  on  $M$ , we consider the subset  $\mathcal{G}_{\eta,u}$  of  $T^*M$  defined by

$$\mathcal{G}_{\eta,u} = \{(x; \eta_x + du_x) : x \in M \text{ such that } du_x \text{ exists}\}.$$

We call the subset  $\mathcal{G} \subset T^*M$  an *overlapping pseudograph* if there exists a closed smooth form  $\eta$  and a semi-concave function  $u$  such that  $\mathcal{G} = \mathcal{G}_{\eta,u}$ . It turns out that  $\mathcal{G}$  fit well to describe unstable manifolds. To describe stable manifolds one considers anti-overlapping pseudographs

$$\check{\mathcal{G}}_{\eta,u} = \{(x; \eta_x - du_x) : x \in M \text{ such that } du_x \text{ exists}\}.$$

Finally an analog of transverse intersection of stable and unstable manifolds  $\mathcal{G}_{\eta,u}$  and  $\check{\mathcal{G}}_{\eta,u'}$  is a property of  $u(x) + u'(x)$  having a *local minimum*. We give a systematic discussion of these facts later.

## 5.2 Forcing relation and shadowing

Here we define forcing relation introduced by Bernard [9].

Let  $\mathcal{G} = \mathcal{G}_{c,u}$  be an overlaying pseudograph, where  $u$  is a semi-concave function. We write  $c(\mathcal{G}) = c$ . We say that

$$\mathcal{G} \vdash_N \mathcal{G}', \quad \text{if} \quad \overline{\mathcal{G}'} \subset \bigcup_{n=1}^N \varphi^n(\mathcal{G}),$$

where  $\varphi$  is the time-1-map of the Hamiltonian flow. We say that  $\mathcal{G} \vdash_N c'$  if there exists a pseudograph  $\mathcal{G}'$  with  $c(\mathcal{G}') = c'$  and  $\mathcal{G} \vdash_N \mathcal{G}'$ . Finally, we say that  $c \vdash c'$  if there exists  $n \in \mathbb{N}$  such that for any pseudograph  $\mathcal{G}$  with  $c(\mathcal{G}) = c$ , we have  $\mathcal{G} \vdash_N c'$ .

The relation  $c \vdash c'$  is transitive, and hence the relation  $c \dashv\vdash c'$ , defined by  $c \vdash c'$  and  $c' \vdash c$ , is an equivalence relation. We call the equivalent classes of this relation the forcing classes. The  $\dashv\vdash$  relation implies existence of various shadowing orbits. In particular, the following hold.

**Theorem 9.** [9, Proposition 0.10]

- If  $c \dashv\vdash c'$ , then there exists a heteroclinic orbit between  $\tilde{\mathcal{A}}(c)$  and  $\tilde{\mathcal{A}}(c')$ .
- Let  $c_i$ ,  $i \in \mathbb{Z}$  be a sequence of cohomologies such that all  $c_i \dashv\vdash c_j$ . Fix a sequence of neighbourhoods  $U_i$  of  $\tilde{\mathcal{M}}(c_i)$ , then there exists an orbit  $(\theta, p)(t)$  of the underlying Hamiltonian flow  $H$ , and  $t_i \in \mathbb{R}$  such that  $(\theta, p)(t_i) \in U_i$ .

In order to connect forcing relation with variational problems we state the following proposition. Let  $L : TM \times \mathbb{T} \rightarrow \mathbb{R}$  be a time-periodic Tonelli Lagrangian (see section 9.1 for definition) and a smooth one form  $\eta : M \rightarrow T^*M$ . Consider a modified action

$$A_\eta(x, t; y, s) = \inf \int_s^t L(\gamma(\tau), \dot{\gamma}(\tau), \tau) - \eta(\dot{\gamma}(\tau)) d\tau,$$

where minimization over the set of absolutely continuous curves  $\gamma : [s, t] \rightarrow M$  with  $\gamma(s) = x$ ,  $\gamma(t) = y \in M$ . Denote by  $\varphi_s^t$  a map from time  $\tau = s$  to  $\tau = t$  for the Euler-Lagrange flow of  $L$ . We have the following

**Proposition 5.1.** [9, Proposition 2.7] Fix an overlapping pseudograph  $\mathcal{G}_{\eta, u}$ , an open set  $U \subset M$  and two times  $s < t$ . Define

$$v(z) = \min_{x \in \bar{U}} u(x) + A_\eta(x, t; z, s),$$

where  $\bar{U}$  is the closure of  $U$ . Let  $V \subset M$  be an open set and let  $N \subset M$  be the set of points, where the minimum is reached in the definition of  $v(z)$  for some  $x \in V$ . If  $\bar{N} \subset U$ , then

$$\overline{\mathcal{G}_{\eta, v|_V}} \subset \varphi_s^t(\mathcal{G}_{\eta, u|_{\bar{N}}})$$

and  $\mathcal{G}_{\eta, u|_{\bar{N}}}$  is a Lipschitz graph above  $\bar{N}$ . In other words, the function  $u$  is differentiable at each point of  $\bar{N}$ , and the map  $x \mapsto du_x$  is Lipschitz on  $\bar{N}$ .

In loose terms, having

inner minima in  $\bar{N} \subset U \implies$  solutions are orbits of the Euler-Lagrange flow.

Moreover, if we have some control on properties of the function  $v(z)$ , then we also have the property that resembles forcing relation. Namely, orbits starting at a restricted

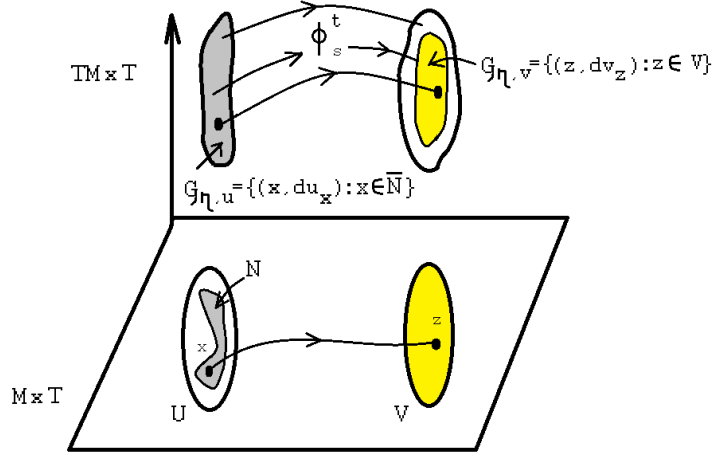


Figure 16: Evolution of pseudographs as solutions to a variational problem

pseudograph  $\mathcal{G}_{\eta, u|_{\bar{N}}}$  flow to contain the closure of a restricted pseudograph  $\overline{\mathcal{G}_{\eta, v|_V}}$ . Fathi [34] showed as  $t - s \rightarrow \infty$  the sum  $v(z)$  converges to certain limits independent of  $u$ . Proper limits are called *weak KAM solutions* (see section 10.1 from precise definitions). In order to connect with the heuristic discussion above the reader can have in mind  $\mathcal{G}_{\eta, u|_{\bar{N}}}$  being a part of the unstable “manifold” of one saddle  $W_{loc}^u(x_i)$  which accumulates to the next one  $W_{loc}^s(x_{i+1})$  under the Euler-Lagrange flow.

We can use Theorem 9 to prove existence of shadowing orbits.

### 5.3 Choice of cohomology classes for global diffusion

Using the forcing relation, we reduce the proof of our main result (Theorem 1) to the following statement.

**Theorem 10.** *Assume that  $H_1 \in \mathcal{U} \subset \mathcal{S}^r$ , in other words, it satisfies all the non-degeneracy conditions we introduced. Then there exists a subset  $\tilde{\Gamma}_* \subset B^2$  with*

$$\text{dist}(\tilde{\Gamma}_*, \Gamma_*) = O(\sqrt{\epsilon}),$$

where  $\text{dist}$  denote the Hausdorff distance with the following property.

*There exists a nonnegative function  $\varepsilon_0 = \varepsilon_0(H_1)$  with  $\varepsilon_0|_{\mathcal{U}} > 0$ , such that for  $\mathcal{V} = \{\epsilon H_1 : H_1 \in \mathcal{U}, 0 < \epsilon < \varepsilon_0\}$ , for a dense set of  $\epsilon H_1 \in \mathcal{V}$ , the cohomologies in  $\tilde{\Gamma}_*$  are all contained in a single forcing class.*

Theorem 9 and Theorem 10 imply existence of diffusion orbits for a dense set of  $\epsilon H_1 \in \mathcal{V}$ . Since existence of diffusion orbits is an open property by the smooth depen-

dence of solutions of ODE on the vector field, we conclude that our main Theorem 1 holds on an open and dense subset  $\mathcal{W}$  of  $\mathcal{V}$ .

**Remark 5.1.** Notice that we do not claim that  $\tilde{\Gamma}_*$  is connected. For example, near double resonances, when we switch from one resonant segment  $\Gamma_j$  to another  $\Gamma_{j+1}$  we might make a jump (see Figure 17 cases b), d), and e)).

Since we only need to prove denseness, Key Theorems 8 and 9 are stated in perturbative setting: given any  $H_1 \in \mathcal{U}$ , there exists arbitrarily small  $C^r$ -perturbation of  $\varepsilon H_1$  such that our theorems hold.

In this section, we first describe the set  $\tilde{\Gamma}_*$ . It is closely related with section 3.5. We write

$$\tilde{\Gamma}_* = \bigcup_j \Gamma_j^{sr} \cup \Gamma_j^{dr},$$

with  $\Gamma_j^{sr} \cap \Gamma_j^{dr} \neq \emptyset$ , and  $\Gamma_j^{dr} \cap \Gamma_{j+1}^{sr} \neq \emptyset$ . Here  $\Gamma_j^{sr}$  corresponds to single resonance and  $\Gamma_j^{dr}$  corresponds to double resonance. We now describe each piece individually.

1. (Single resonance) In the single resonance regime, we choose  $\Gamma_j^{sr}$  to be a passage segment (see (12)). Key Theorem 8 states that all cohomogies in  $\Gamma_j^{sr}$  are in the same forcing class.

In order to prove forcing-equivalence of all  $\tilde{\Gamma}_*$  we also need  $\Gamma_j^{dr}$  and the nearby  $\Gamma_j^{sr}$  is disjoint. This is done by making a  $\sqrt{\varepsilon}$  modification to  $\Gamma_j^{sr}$ , see section D.

2. (Double resonance) In the double resonance regime, the choice of  $\Gamma_j^{dr}$  depends on the homologies  $h$  and  $h'$  associated with this double resonance, as well as the direction of diffusion (going across or turning the corner). See Figure 17 for an illustration of all cases.

- If  $h$  is simple and critical, and the diffusion is going across: we define

$$\Gamma_{h,s}^{0,E_0} = \bigcup_{0 \leq E \leq E_0} c_h(E) \cup \bigcup_{0 \leq \lambda \leq 1} c_h^\lambda, \quad \Gamma_h^{E_0, \bar{E}} = \bigcup_{E_0 \leq E \leq \bar{E}} c_h(E), \quad (10)$$

where the function  $c_h(E)$  and  $c_h^\lambda$  are defined in sections 4.2.2 and 4.2.3. We choose

$$\Gamma_j^{dr} = \Gamma_h^{E_0, \bar{E}} \cup (\Gamma_{h,s}^{0,E_0} \cup \Gamma_{-h,s}^{0,E_0}) \cup \Gamma_{-h}^{E_0, \bar{E}}.$$

- If  $h$  is simple and non-critical, the choice of cohomology is identical to the critical case.

The Aubry sets of the above cohomologies are localized in the high energy cylinders and a simple low energy cylinder. For low energy, the diffusion orbit is going from the simple cylinder of homology  $h$  to the simple cylinder of homology  $-h$ . See Figure 17, a) for both cases.

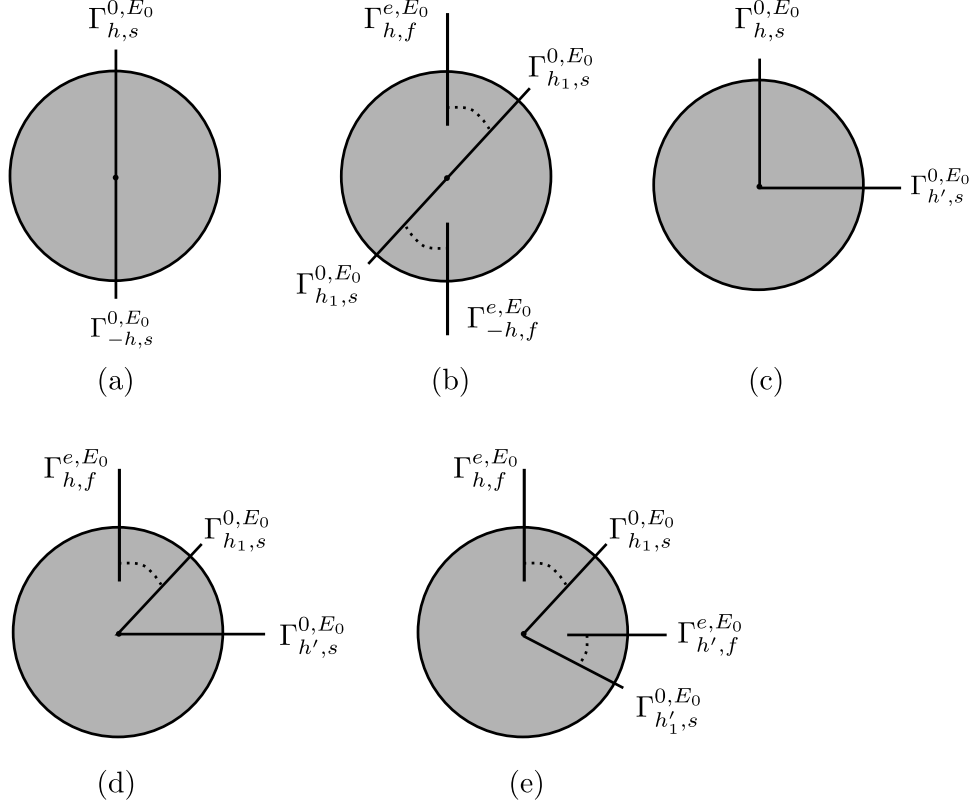


Figure 17: *Choice of cohomology classes for double resonance:* (a) simple homology, across resonance; (b) non-simple, across; (c) simple to simple; (d) non-simple to simple; (e) non-simple to non-simple.

- If  $h$  is non-simple, with decomposition  $h = n_1 h_1 + n_2 h_2$  into simple  $h_1, h_2$ . The diffusion is going across. Let  $e > 0$  be a small number depending only on  $H^s$  (and hence depends on  $H_1, \Gamma^*, r$ ) to be determined later. We define

$$\Gamma_{h,f}^{e,\bar{E}} = \bigcup_{e \leq E \leq E_0} c_h(E) \quad (11)$$

and let

$$\Gamma_{h_1,s}^{0,E_0} = \bigcup_{0 \leq E \leq E_0} c_{h_1}^e(E) \cup \bigcup_{0 \leq \lambda \leq 1} c_{h_1}^{e,\lambda},$$

where the functions  $c_{h_1}^e(E)$  and  $c_{h_1}^{e,\lambda}$  are defined in section 4.3. These cohomologies corresponds to a simple critical loop for localization purposes, and enjoy the additional properties specified in section 4.3 (see Propositions 4.2 and 4.3).

We choose

$$\Gamma_j^{dr} = \Gamma_{h,f}^{e,\bar{E}} \cup (\Gamma_{h_1,s}^{0,E_0} \cup \Gamma_{-h_1,s}^{0,E_0}) \cup \Gamma_{-h,f}^{e,\bar{E}}.$$

Diffusion orbits are jumping from the flower cylinder  $\mathcal{M}_{h,\varepsilon}^{e,\bar{E}}$  of  $h$  to the simple cylinder  $\mathcal{M}_{h_1,\varepsilon}^{0,E_0}$  of  $h_1$ , which is connected to the simple cylinder  $\mathcal{M}_{-h_1,\varepsilon}^{0,E_0}$  of  $-h_1$ , and finally back to the flower cylinder  $\mathcal{M}_{-h,\varepsilon}^{e,\bar{E}}$  of  $-h$ . See Figure 17, b).

- If both  $h$  and  $h'$  are simple, and diffusion orbits are turning the corner: we define

$$\Gamma_j^{dr} = \Gamma_h^{E_0,\bar{E}} \cup (\Gamma_{h,s}^{0,E_0} \cup \Gamma_{-h',s}^{0,E_0}) \cup \Gamma_{-h'}^{E_0,\bar{E}}.$$

We are jumping from a simple cylinder  $\mathcal{M}_{h,\varepsilon}^{0,E_0}$  of homology  $h$  to a simple cylinder  $\mathcal{M}_{h',\varepsilon}^{0,E_0}$  of homology  $h'$ . See Figure 17, c).

- If  $h$  is non-simple with decomposition  $h = n_1 h_1 + n_2 h_2$  into simple, and we are turning to a simple  $h'$ : we define

$$\Gamma_j^{dr} = \Gamma_{h,f}^{e,\bar{E}} \cup \Gamma_{h_1,s}^{0,E_0} \cup \Gamma_{h',s}^{0,E_0} \cup \Gamma_{h',f}^{E_0,\bar{E}}.$$

We jump from the non-simple homology  $h$  to a simple  $h_1$ , then to the simple  $h'$ . For the case  $h'$  is non-simple and  $h$  is simple, simply switch  $h$  and  $h'$  in the above definition. See Figure 17, d).

- If  $h$  and  $h'$  are both non-simple with decompositions  $h = n_1 h_1 + n_2 h_2$  and  $h' = m_1 h'_1 + m_2 h'_2$  into simple ones, and we are turning the corner: we define

$$\Gamma_j^{dr} = \Gamma_{h,f}^{e,\bar{E}} \cup \Gamma_{h_1,s}^{0,E_0} \cup \Gamma_{h'_1,s}^{0,E_0} \cup \Gamma_{h',f}^{e,\bar{E}}.$$

Diffusion jump from the non-simple  $h$  to a simple  $h_1$ , then to a simple  $h'_1$  and jumps to the non-simple  $h'$ . See Figure 17, e).

3. Key Theorem 9 states that it is possible to diffuse along high energy cylinder, flower cylinder or simple cylinder. In the variational language, it asserts the cohomologies in  $\Gamma_{h,f}^{e,\bar{E}}$  (resp.  $\Gamma_{h,s}^{0,E_0}$ ,  $\Gamma_h^{E_0,\bar{E}}$ ) are equivalent. They correspond to the solid segments in Figure 17.
4. Key Theorem 10 covers the crucial “jump”. It asserts that there exists some  $c \in \Gamma_{h,f}^{e,E_0}$  and  $c' \in \Gamma_{h,s}^{0,E_0}$ , such that  $c \dashv\vdash c'$ . The jump corresponds to the dotted curves in Figure 17. As a consequence, all cohomologies in  $\Gamma_j^{dr}$  are equivalent.

## 6 Equivalent forcing classes

In this section we discuss equivalent forcing classes in three different regimes:

- in a single resonance along one cylinder,
- in a single resonance along cylinders of the same homology class,
- in a double resonance between kissing cylinders.

### 6.1 Equivalent forcing class along single resonances

In the single resonant we choose  $\Gamma_j^{sr} \subset \Gamma_j$  as a passage segment. More precisely,

$$\Gamma_j^{sr} = \{(p_*^s(p^f), p^f); p^f \in [a_-, a_+]\} \subset \Gamma_j, \quad (12)$$

where  $[a_-^{(j)}, a_+^{(j)}]$  is as in subsection 3.1. Now we omit superscript  $j$  for brevity. We have  $[a_-, a_+] = \bigcup_{i=1}^{s-1} [a_-^i, a_+^i]$ , and by Key Theorems 4 and 5, for each  $c \in \Gamma_j^{sr}$ , the Aubry set  $\tilde{\mathcal{A}}(c)$  is contained in one of the NHICs, and is a graph over the  $\theta^f$  component.

**Key Theorem 8.** *Assume that the Hamiltonian  $H_\epsilon = H_0 + \epsilon H_1$  satisfies the non-degeneracy conditions [G0]-[G2], then there exists arbitrarily small perturbation  $\epsilon H_1'$  of  $\epsilon H_1$ , such that for the Hamiltonian  $H_\epsilon'' = H_0 + \epsilon H_1''$ ,  $\Gamma_i^{sr}$  is contained in a single forcing class.*

Diffusion along  $\Gamma_i^{sr}$  contains three different phenomena: diffusing inside of the cylinder, climbing up a cylinder using normal hyperbolicity outside of the cylinder, and jumping from one cylinder to the other. We have the following definitions:

- (*passage values*) We say that  $c \in \Gamma_1 \subset \Gamma_i^{sr}$ , if  $\tilde{\mathcal{N}}(c)$  is contained in only one cylinder, has only one static class, and the projection onto  $\theta^f$  component is not the whole circle. Due to a result of Mather [54] (and in the forcing setting, [9]),  $c$  is in the interior of its forcing class.
- (*bifurcation values*) We say that  $c \in \Gamma_2 \subset \Gamma_i^{sr}$ , if  $\tilde{\mathcal{A}}(c)$  has exactly two static classes and each contained in one NHIC. In this case we would like to jump from one cylinder to another.
- (*invariant curve values*) We say that  $c \in \Gamma_3 \subset \Gamma_i^{sr}$ , if  $\tilde{\mathcal{A}}(c)$  is contained in a single cylinder, but the projection onto  $\theta^f$  component is the whole circle. In this case it is impossible to move within the cylinder, the normal hyperbolicity will be used.



**Theorem 11.** *Assume that the Hamiltonian  $H_\epsilon = H_0 + \epsilon H_1$  satisfies the nondegeneracy conditions [G0]-[G2], then there exists an arbitrarily small perturbation  $\epsilon H'_1$  of  $\epsilon H_1$ , such that for  $H'_\epsilon = H_0 + \epsilon H'_1$ ,  $\Gamma_2$  is finite, and*

$$\Gamma_i^{sr} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \quad (13)$$

The diffusion for both bifurcation values  $\Gamma_2$  and invariant curve values  $\Gamma_3$  requires additional transversalities. In the case that  $\tilde{\mathcal{A}}(c)$  is an invariant circle, this transversality condition is equivalent to the transversal intersection of the stable and unstable manifolds. This condition can be phrased in terms of the barrier functions. Recall that if we need to emphasise dependence of the Aubry  $\mathcal{A}(c)$  and Mañe  $\mathcal{N}(c)$  sets on the associated Hamiltonian  $H$  we write  $\mathcal{A}_H(c)$  and  $\mathcal{N}_H(c)$ .

In the bifurcation set  $\Gamma_2$ , the Aubry set  $\mathcal{A}(c)$  has exactly two static classes. In this case the Mañe set  $\mathcal{N}(c) \supsetneq \mathcal{A}(c)$ . Let  $\theta_0$  and  $\theta_1$  be contained in each of the two static classes of  $\mathcal{A}(c)$ , we define

$$b_{H_\epsilon, c}^+(\theta) = h_{H_\epsilon, c}(\theta_0, \theta) + h_{H_\epsilon, c}(\theta, \theta_1)$$

and

$$b_{H_\epsilon, c}^-(\theta) = h_{H_\epsilon, c}(\theta_1, \theta) + h_{H_\epsilon, c}(\theta, \tilde{\theta}_0),$$

where  $h_{H_\epsilon, c}$  is the Peierls barrier for cohomology class  $c$  associated to the Hamiltonian  $H_\epsilon$ . It is defined in section 10. Let  $\Gamma_2^*$  be the set of bifurcation  $c \in \Gamma_2$  such that the minima of each  $b_{H_\epsilon, c}^+$  and  $b_{H_\epsilon, c}^-$  outside of  $\mathcal{A}_{H_\epsilon}(c)$  are totally disconnected. In other words, these minima correspond to heteroclinic orbits connecting different components of the Aubry set  $\mathcal{A}_{H_\epsilon}(c)$  and heteroclinic orbits  $\mathcal{N}_{H_\epsilon}(c) \setminus \mathcal{A}_{H_\epsilon}(c)$  form a not empty and totally disconnected set.

Since the Aubry and the Mañe sets are *symplectic invariants* [10], it suffices to prove these properties *using a convenient canonical coordinates, e.g. normal forms*.

In the case  $c \in \Gamma_3$ , we have

$$\tilde{\mathcal{N}}(c) = \tilde{\mathcal{I}}(c, u) = \tilde{\mathcal{A}}(c)$$

and it is an invariant circle<sup>13</sup>. We first consider the covering

$$\begin{aligned} \xi : \mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\ \theta = (\theta^f, \theta^s) &\longmapsto \xi(\theta) = (\theta^f, 2\theta^s). \end{aligned}$$

This covering lifts to a symplectic covering

$$\begin{aligned} \Xi : T^*\mathbb{T}^2 &\longrightarrow T^*\mathbb{T}^2 \\ (\theta, p) = (\theta, p^f, p^s) &\longmapsto \Xi(\theta, p) = (\xi(\theta), p^f, p^s/2), \end{aligned}$$

---

<sup>13</sup>see Section 9.4 for definition of  $\tilde{\mathcal{I}}(c, u)$ , which implies  $\tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{I}}(c, u) \subset \tilde{\mathcal{N}}(c)$ .

and we define the lifted Hamiltonian  $\tilde{H}_\varepsilon = H_\varepsilon \circ \Xi$ . It is known that

$$\tilde{\mathcal{A}}_{\tilde{H}}(\tilde{c}) = \Xi^{-1}(\tilde{\mathcal{A}}_H(c))$$

where  $\tilde{c} = \xi^*c = (c^f, c^s/2)$ . On the other hand, the inclusion

$$\tilde{\mathcal{N}}_{\tilde{H}}(\tilde{c}) \supset \Xi^{-1}(\tilde{\mathcal{N}}_H(c)) = \Xi^{-1}(\tilde{\mathcal{A}}_H(c))$$

is not an equality for  $c \in \Gamma_3$ . More precisely, for  $c \in \Gamma_3(\epsilon)$ , the set  $\tilde{\mathcal{A}}_{\tilde{H}}(\tilde{c})$  is the union of two circles, while  $\tilde{\mathcal{N}}_{\tilde{H}}(\tilde{c})$  contains heteroclinic connections between these circles. Similarly to the case of  $\Gamma_2$ , we choose a point  $\theta_0$  in the projected Aubry set  $\mathcal{A}_{H_\varepsilon}(c)$ , and consider its two preimages  $\tilde{\theta}_0$  and  $\tilde{\theta}_1$  under  $\xi$ . We define

$$b_{\tilde{H}_\varepsilon, c}^+(\theta) = h_{\tilde{H}_\varepsilon, c}(\tilde{\theta}_0, \theta) + h_{\tilde{H}_\varepsilon, c}(\theta, \tilde{\theta}_1)$$

and

$$b_{\tilde{H}_\varepsilon, c}^-(\theta) = h_{\tilde{H}_\varepsilon, c}(\tilde{\theta}_1, \theta) + h_{\tilde{H}_\varepsilon, c}(\theta, \tilde{\theta}_0)$$

where  $h_{\tilde{H}_\varepsilon, c}$  is the Peierl's barrier associated to  $\tilde{H}_\varepsilon$ .  $\Gamma_3^*(\epsilon)$  is then the set of cohomologies  $c \in \Gamma_3(\epsilon)$  such that the minima of each of the functions  $b_{\tilde{H}_\varepsilon, c}^\pm$  located outside of the Aubry set  $\mathcal{A}_{\tilde{H}_\varepsilon}(\tilde{c})$  are totally disconnected. In other words,  $\tilde{\mathcal{N}}_{\tilde{H}_\varepsilon}(\tilde{c}) \setminus \mathcal{A}_{\tilde{H}_\varepsilon}(\tilde{c})$  is not empty and totally disconnected. Since the Aubry and the Mañé sets are *symplectic invariants* [10], it suffices to prove these properties *using a convenient canonical coordinates, e.g. normal forms*.

We can perform an additional perturbation such that the above transversality condition is satisfied.

**Theorem 12.** *Let  $H'_\varepsilon = H_0 + \epsilon H'_1$  be the Hamiltonian from Theorem 11. Then there exists arbitrarily small perturbation  $\epsilon H''_1$  of  $\epsilon H'_1$ , preserving all Aubry sets  $\mathcal{A}_{\tilde{H}_\varepsilon}(\tilde{c})$  with  $c \in \Gamma_j^{sr}$ , such that for  $H''_\varepsilon = H_0 + \epsilon H''_1$  we have  $\Gamma_2 = \Gamma_2^*$  and  $\Gamma_3 = \Gamma_3^*$ . Moreover,  $\Gamma_j^{sr} = \Gamma_1 \cup \Gamma_2^* \cup \Gamma_3^*$  is contained in a single forcing class.*

Clearly, Key Theorem 8 follows from Theorem 12. The proof of this theorem relies on Key Theorems 4 and 5 about localization of Aubry sets inside of a proper cylinder and Lipschitz graph properties of these sets over  $\mathbb{T}^2 \ni (\theta^f, t)$ . Recall that Key Theorems 4 and 5 are proven in the same way as Theorems 5.1 and 5.2 in [13]. This Theorem is analogous to Theorem 6.4 in [13].

## 6.2 Equivalent forcing class along cylinders of the same homology class

The chosen cohomology classes  $\Gamma_i^{dr}$  in the double resonance region consists of possibly several connected components. Each component is either  $\Gamma_{h,s}^{0, \bar{E}}$  or  $\Gamma_{h,s}^{0, E_0}$  for a simple

homology  $h$ , or  $\Gamma_{h,f}^{e,\bar{E}}$  for a non-simple homology. The following key theorem establishes forcing equivalence for each of the connected components.

Recall that in section 4.2.3 for each homology class  $h \in H_1(\mathbb{T}^s, \mathbb{Z})$  under the condition [A0] we have existence of only three possibilities for  $\gamma_h^0$ :

- If  $h$  is simple and non critical,  $\mathcal{A}_{H^s}(\bar{c}_h(0)) = \gamma_h^0 \cup \{0\}$ .
- If  $h$  is simple and critical,  $\mathcal{A}_{H^s}(\bar{c}_h(0)) = \gamma_h^0$  and  $\gamma_h^0$  contains 0.
- If  $h$  is non-simple and  $h = n_1 h_1 + n_2 h_2$  is decomposition into simple,  $\mathcal{A}_{H^s}(\bar{c}_h(0)) = \gamma_{h_1}^0 \cup \gamma_{h_2}^0$ .

**Key Theorem 9.** *With notations above consider the perturbed Hamiltonian  $H_\epsilon = H_0 + \epsilon H_1$ , a strong double resonance and associated integer homology classes  $h, h_1, h'_1 \in H_1(\mathbb{T}^s, \mathbb{Z})$ . Suppose in each item listed below the corresponding conditions hold and that the parameter  $\epsilon$  is such that Key Theorem 3 applies. Then there exists an arbitrary small localized  $C^r$  perturbation  $\epsilon \Delta H_1$  of  $H_\epsilon$  such that for the Hamiltonian  $H'_\epsilon = H_\epsilon + \epsilon \Delta H_1$  an appropriate family of cohomologies belongs to a single forcing class.*

- (high energy) *If  $h$  is simple and satisfies the conditions [DR1]-[DR3]. Then for  $H'_\epsilon$  the family of cohomologies  $\Gamma_h^{E_0, \bar{E}}$  is contained in a single forcing class.*
- (high energy) *If  $h$  is non-simple homology and satisfies the conditions [DR1]-[DR3]. Then for  $H'_\epsilon$  the family of cohomologies  $\Gamma_{h,f}^{e, \bar{E}}$  is contained in a single forcing class.*
- (low energy) *If  $h_1, h'_1$  are simple and critical homologies and satisfy the conditions [DR1]-[DR3], and conditions [A1]-[A4] of Key Theorem 3. Then for  $H'_\epsilon$  the family of cohomologies*

$$\Gamma_{h_1,s}^{0,E_0} \cup \Gamma_{h'_1,s}^{0,E_0} \cup \Gamma_{-h_1,s}^{0,E_0} \cup \Gamma_{-h'_1,s}^{0,E_0}$$

*is contained in a single forcing class.*

- (low energy) *If  $h$  is simple and non critical homology and satisfies the conditions [DR1]-[DR3] for all energies  $[-\delta, E_0 + \delta]$ . Then for  $H'_\epsilon$  the family of cohomologies  $\Gamma_h^{0,E_0} \cup \Gamma_{-h}^{0,E_0}$  is contained in a single forcing class.*

Recall that by Proposition B.4 near a double resonance after a proper rescaling  $S$  and a canonical change of coordinates  $\Phi_\epsilon$  the perturbed system has the form

$$H_\epsilon^s(\theta^s, I^s, \tau) = \mathcal{S}(H_\epsilon \circ \Phi_\epsilon) = \frac{H_0(p_0)}{\epsilon} + K(I^s) - U(\theta^s) + \sqrt{\epsilon} P(\theta^s, I^s, \tau),$$

$$\theta^s \in \mathbb{T}^s, \quad I^s \in \mathbb{R}^2, \quad \tau \in \sqrt{\varepsilon} \mathbb{T}.$$

We denote

$$\bar{\bar{H}}_\varepsilon^s = \mathcal{S}((H_\varepsilon + \Delta H_1) \circ \Phi_\varepsilon) = H^s + \sqrt{\varepsilon} \bar{\bar{P}}.$$

**Remark 6.1.** *It turns out that supports of perturbation  $\Delta H_1$  of  $H_\varepsilon$  (resp.  $\sqrt{\varepsilon} \bar{\bar{P}}$  of  $\sqrt{\varepsilon} P$ ) are localized. We distinguish simple critical homologies  $h_1$  and  $h'_1$  in item 3, because we need additional information about supports of these perturbations.*

The proof of this Theorem is somewhat similar to the proof of Key Theorem 8. In particular, the perturbation  $\bar{\bar{H}}_\varepsilon^s = H^s + \sqrt{\varepsilon} \bar{\bar{P}}$  consists of two steps.

STEP 1. Perturb  $H_\varepsilon^s$  to

$$\bar{H}_\varepsilon^s = H^s + \sqrt{\varepsilon} \bar{P}$$

such that each of four the families of cohomologies

- (high energy simple)  $\Gamma_h^{E_0, \bar{E}}$ ,
- (high energy non-simple)  $\Gamma_{h,f}^{e, \bar{E}}$
- (low energy simple, critical)  $\Gamma_{h_1,s}^{0,E_0} \cap \Gamma_{-h_1,s}^{0,E_0}$  and  $\Gamma_{h'_1,s}^{0,E_0} \cap \Gamma_{-h'_1,s}^{0,E_0}$
- (high energy simple, non-critical)  $\Gamma_h^{0,E_0}$  and  $\Gamma_{-h}^{0,E_0}$

consists of only three types, defined be analogy with (13).

- (*passage values*) Let  $\Gamma_{\bullet,1}^{\bullet,*}$  be the union of the following two subsets.
  - The first subset is the set of all  $c \in \Gamma$  such that  $\tilde{\mathcal{N}}_{H_\varepsilon}(c)$  is contained in only one cylinder, and the image of  $\tilde{\mathcal{N}}_{H_\varepsilon}(c)$  under the projection to  $\gamma_h^E$  is not the whole curve.
  - The second subset is the set of all  $c \in \Gamma$  such that  $\tilde{\mathcal{A}}_{H_\varepsilon}(c) \subset o_\varepsilon$ .
- (*bifurcation values*) Let  $\Gamma_{\bullet,2}^{\bullet,*}$  be the set of all  $c \in \Gamma$  such that  $\tilde{\mathcal{A}}_{H_\varepsilon}(c)$  has exactly two static class, each contained in an invariant cylinder.
- (*invariant curve values*) Let  $\Gamma_{\bullet,3}^{\bullet,*}$  be the set of all  $c \in \Gamma$  such that  $\tilde{\mathcal{A}}_{H_\varepsilon}(c)$  is contained in a single cylinder, and the projection of  $\tilde{\mathcal{A}}_{H_\varepsilon}(c)$  to  $\gamma_h^E$  is onto. In other words, the intersection of  $\tilde{\mathcal{A}}_{H_\varepsilon}(c)$  with the section  $\{t = 0\}$  is an invariant curve.

Moreover, if the slow Hamiltonian  $H^s$  satisfies conditions of Key Theorems 2 and 3, then by Key Theorem 7 the Aubry sets of  $H_\varepsilon^s$  satisfy Mather's projected graph theorem. The claim below improves Key Theorem 7 for a generic perturbation  $P$ . In the item by item setting of Key Theorem 9 there exists an arbitrary small localized  $C^r$  perturbation  $\bar{P}$  of  $P$  such that for the Hamiltonian

$$\bar{H}_\varepsilon^s = H^s + \sqrt{\varepsilon} \bar{P}$$

an appropriate family of cohomologies consists of only three aforementioned classes with the second class being finite. Here is the formal claim:

**Theorem 13.** • *(high energy) If  $h$  is simple and satisfies the conditions [DR1]-[DR3]. Then there is an arbitrary  $C^r$  small perturbation  $\bar{P}$  of  $P$  such that  $\sqrt{\varepsilon}(\bar{P} - P)$  is localized in a neighborhood of the normally hyperbolic weakly invariant cylinders*

$$\bigcup_{j=0}^{N-1} \mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$$

*such that for the Hamiltonian  $\bar{H}_\varepsilon^s$  the families of cohomologies satisfy*

$$\Gamma_h^{E_0, \bar{E}} = \Gamma_{h,1}^{E_0, \bar{E}} \cup \Gamma_{h,2}^{E_0, \bar{E}} \cup \Gamma_{h,3}^{E_0, \bar{E}}$$

*where  $\Gamma_{h,2}^{E_0, \bar{E}}$  is finite.*

- *(high energy) If  $h$  be non-simple homology and satisfies the conditions [DR1]-[DR3]. Then there is an arbitrary  $C^r$  small perturbation  $\bar{P}$  of  $P$  such that  $\sqrt{\varepsilon}(\bar{P} - P)$  is localized in a neighborhood of the invariant cylinders*

$$\mathcal{M}_{h,\varepsilon}^{e, E_0} \cup \bigcup_{j=0}^{N-1} \mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$$

*such that for the Hamiltonian  $\bar{H}_\varepsilon^s$  the families of cohomologies satisfy*

$$\Gamma_{h,f}^{e, \bar{E}} = \Gamma_{h,1}^{e, \bar{E}} \cup \Gamma_{h,2}^{e, \bar{E}} \cup \Gamma_{h,3}^{e, \bar{E}}$$

*where  $\Gamma_{h,2}^{e, \bar{E}}$  is finite.*

- *(low energy) If  $h_1, h'_1$  are simple and critical homologies and satisfy the conditions [DR1]-[DR3], and conditions [A1]-[A4] of Key Theorem 3. Then there is an arbitrary  $C^r$  small perturbation  $\bar{P}$  of  $P$  such that  $\sqrt{\varepsilon}(\bar{P} - P)$  is localized in a neighborhood of the normally hyperbolic weakly invariant cylinder*

$$\mathcal{M}_{h_1,\varepsilon}^{E_0,s} \cup \mathcal{M}_{h'_1,\varepsilon}^{E_0,s},$$

such that for the Hamiltonian  $\tilde{H}_\epsilon^s$  the families of cohomologies satisfy

$$\Gamma_{h_1,s}^{0,E_0} \cup \Gamma_{-h_1,s}^{0,E_0} = \Gamma_{\pm h_1,1}^{0,E_0} \cup \Gamma_{\pm h_1,2}^{0,E_0} \cup \Gamma_{\pm h_1,3}^{0,E_0}$$

and

$$\Gamma_{h'_1,s}^{0,E_0} \cup \Gamma_{-h'_1,s}^{0,2E_0} = \Gamma_{\pm h'_1,1}^{0,E_0} \cup \Gamma_{\pm h'_1,2}^{0,E_0} \cup \Gamma_{\pm h'_1,3}^{0,2E_0}$$

where  $\Gamma_{\pm h_1,2}^{0,E_0}$  and  $\Gamma_{\pm h'_1,2}^{0,E_0}$  are finite.

- (low energy) If  $h$  is simple and non critical homology and satisfies the conditions [DR1]-[DR3] for all energies  $[-\delta, E_0 + \delta]$ . Then there is an arbitrary  $C^r$  small perturbation  $\bar{P}$  of  $P$  such that  $\sqrt{\epsilon}(\bar{P} - P)$  is localized in a neighborhood of the normally hyperbolic weakly invariant cylinders

$$\mathcal{M}_{h,\epsilon}^{0,E_0} \cup \mathcal{M}_{-h,\epsilon}^{0,E_0}$$

such that

$$\Gamma_{h,s}^{0,E_0} \cup \Gamma_{-h,s}^{0,E_0} = \Gamma_{\pm h,1}^{0,\bar{E}} \cup \Gamma_{h,2}^{0,\bar{E}} \cup \Gamma_{h,3}^{0,\bar{E}},$$

where  $\Gamma_{\pm h,2}^{0,\bar{E}}$  are finite.

Existence of diffusion in both cases bifurcation values  $\Gamma_{\bullet,2}^{\bullet,*}$  and invariant curve values  $\Gamma_{\bullet,3}^{\bullet,*}$  require additional transversalities. In the case that  $\tilde{\mathcal{A}}(c)$  is an invariant circle, this transversality condition is equivalent to the transversal intersection of the stable and unstable manifolds. This condition can be phrased in terms of the barrier functions.

Similar to the case of single resonance, we need to make some further definitions.

Let  $\Gamma_{\bullet,2}^{\bullet,*}$  be the set of bifurcation  $c \in \Gamma_{\bullet,2}^{\bullet,*}$  such that the set  $\mathcal{N}_{\tilde{H}_\epsilon^s}(c) \setminus \mathcal{A}_{\tilde{H}_\epsilon^s}(c)$  is totally disconnected.

To make an analogous definition for  $\Gamma_3$ , we need to consider a covering space. First we define a covering map of the slow torus  $\mathbb{T}^s$ :

$$\bar{\xi} : \mathbb{T}^s \longrightarrow \mathbb{T}^s, \quad \bar{\xi}(\varphi^{ss}, \varphi^{sf}) = (2\varphi^{ss}, \varphi^{sf}).$$

The covering map induces a covering map of the cotangent bundle

$$\bar{\Xi} : T^*\mathbb{T}^s \longrightarrow T^*\mathbb{T}^s, \quad \bar{\Xi}(\varphi, p^{ss}, p^{sf}) = (\bar{\xi}(\varphi), p^{ss}/2, p^{sf}).$$

Let  $L : T^*\mathbb{T}^2 \longrightarrow T^*\mathbb{T}^s$  be the linear coordinate change associated with the double resonance (see section 3.2), then the map  $\Xi := L \circ \bar{\Xi} \circ L^{-1}$  defines a symplectic double covering map. The Hamiltonian  $H_\epsilon$  lifts to a Hamiltonian  $\tilde{H}_\epsilon$  under the double cover.

We define the set  $\Gamma_{\bullet,3}^{\bullet,*}$  as the set  $c \in \Gamma_{\bullet,3}^{\bullet,*}$  such that the set

$$\mathcal{N}_{\tilde{H}_\epsilon}(c) \setminus \mathcal{A}_{\tilde{H}_\epsilon}(c)$$

is totally disconnected.

**Theorem 14.** Assume that for the Hamiltonian  $\bar{H}_\varepsilon$  and an integer homology class  $h \in H_1(\mathbb{T}^s, \mathbb{Z})$  the conditions of Theorem 13 are satisfied. Then there exists an additional arbitrarily  $C^r$  small localized perturbation  $\sqrt{\varepsilon} \bar{\bar{P}}$  of  $\sqrt{\varepsilon} \bar{P}$  such that for all cohomology classes  $c \in \Gamma$  the Aubry sets  $\tilde{\mathcal{A}}(c)$  of  $\bar{H}_\varepsilon^s$  and the Hamiltonian

$$\bar{\bar{H}}_\varepsilon^s = H^s + \sqrt{\varepsilon} \bar{\bar{P}}$$

coincide with those of  $H_\varepsilon^s$  and for  $\bar{\bar{H}}_\varepsilon^s$  these sets satisfy

1. (high energy) If  $h$  is simple, then  $\Gamma_{h,2}^{E_0, \bar{E}} = \Gamma_{h,2}^{E_0, \bar{E};*}$  and  $\Gamma_{h,3}^{E_0, \bar{E}} = \Gamma_{h,3}^{E_0, \bar{E};*}$ .
2. (high energy) If  $h$  is non-simple, then  $\Gamma_{h,2}^{e, \bar{E}} = \Gamma_{h,2}^{e, \bar{E};*}$  and  $\Gamma_{h,3}^{e, \bar{E}} = \Gamma_{h,3}^{e, \bar{E};*}$ .
3. (low energy) If  $h_1$  and  $h'_1$  simple and critical, then  $\Gamma_{\pm h_1, 2}^{0, E_0} = \Gamma_{\pm h_1, 2}^{0, E_0;*}$  and  $\Gamma_{\pm h_1, 3}^{0, E_0} = \Gamma_{\pm h_1, 3}^{0, E_0;*}$  as well as  $\Gamma_{\pm h'_1, 2}^{0, E_0} = \Gamma_{\pm h'_1, 2}^{0, E_0;*}$  and  $\Gamma_{\pm h'_1, 3}^{0, E_0} = \Gamma_{\pm h'_1, 3}^{0, E_0;*}$ .
4. (low energy) If  $h$  is simple and non-critical, then  $\Gamma_{\pm h, 2}^{0, E_0} = \Gamma_{\pm h, 2}^{0, E_0;*}$  and  $\Gamma_{\pm h, 3}^{0, E_0} = \Gamma_{\pm h, 3}^{0, E_0;*}$ .

Furthermore, in each case the set  $\Gamma_{\bullet}^{*,*} = \Gamma_{\bullet,1}^{*,*} \cup \Gamma_{\bullet,2}^{*,*} \cup \Gamma_{\bullet,3}^{*,*}$  is contained in a single forcing class.

### 6.3 Equivalent forcing class between kissing cylinders

Let  $h = n_1 h_1 + n_2 h_2$  be a non-simple homology class,  $h_1$  and  $h_2$  are the corresponding simple ones. We have proved that the cohomologies  $\Gamma_{h,f}^{e, E_0}$  (resp.  $\Gamma_{h_1,s}^{0, E_0}$ ) is contained in a single respective forcing class. To finally conclude our proof, we will show that  $\Gamma_{h,f}^{e, E_0}$  and  $\Gamma_{h_1,s}^{0, E_0}$  are equivalent to each other.

Recall that relation between cohomology of  $H_\varepsilon$  and its rescaling  $H_\varepsilon^s$  is given by  $c_h(E) = p_0 + \bar{c}_h(E)\sqrt{\varepsilon}$  and  $c_{h_1}(E) = p_0 + \bar{c}_{h_1}(E)\sqrt{\varepsilon}$  (see (9)). Then in Propositions 4.2 and 4.3 we modify the latter family of cohomologies  $\bar{c}_{h_1}(E)$  relative to  $\bar{c}_h(E)$ . With these notations we have the following statement.

**Key Theorem 10.** Given  $H_1 \in \mathcal{U}$ , there exists  $\varepsilon_0 = \varepsilon_0(H_0, H_1, \Gamma^*, r)$  and  $e_0 = e_0(H_0, H_1, \Gamma^*, r) > 0$ , such that the following hold. Let  $\bar{H}_\varepsilon$  be the perturbed Hamiltonian as in Key Theorem 9. For all  $0 < \varepsilon < \varepsilon_0$ , there exists  $e_0 \leq E_1, E_2 \leq 2e_0$  such that

$$c_h(E_1) \dashv\vdash c_{h_1}^{e_0/2}(E_2).$$

**Remark 6.2.** Notice that on Figure 17 the only type of jump needed is the “jump” from non-simple homology to simple critical one occurring in cases (b), (d), and (e). In all other cases we do not need a “jump”.

We will prove the above theorem by proving a specific constrained variational problem has a nondegenerate minimum. The minimal of this variational problem corresponds to a heteroclinic orbit between the Aubry sets  $\mathcal{A}_{H_\epsilon}(c_h(E_1))$  and  $\mathcal{A}_{H_\epsilon}(c_{h_1}(E_2))$ . We will first define a variational problem for the slow mechanical system  $H^s$ , then for the perturbed slow system  $H_\epsilon^s$ , and finally define it for  $H_\epsilon$  using the associated coordinate changes. This is done in Section 12. Now we outline content of the rest of the paper.

In section 7 the main result is Theorem 15, which along with Theorem 4.1 [13] imply Key Theorem 1.

In section 8 we prove Key Theorem 3. Proof of Key Theorem 2 is discussed above in section 3.2.

Key Theorems 4 and 5 are essentially proved in [13] (see Theorem 5.1 and 5.2).

In section 9 we discuss our diffusion mechanism and basic notions of weak KAM theory: Tonelli Lagrangians/Hamiltonians, overlapping pseudographs, Lax-Oleinik mapping, the Aubry, Mather, Mañe sets, Mather  $\alpha$  and  $\beta$ -functions. Finally, we define forcing relation proposed by Bernard [9].

In section 10 we define basic notions of Mather theory such as barrier functions, the (projected) Aubry and the (projected) Mañe sets, uniform families of Lagrangians/Hamiltonians, super-differentials and semi-continuity of barrier functions.

In section 11 we prove Key Theorems 6 (localization of Aubry and Mañe sets), 7 (graph theorem) and 9 (about forcing relation) along the same homology class.

As we pointed out in section 6 Key Theorem 8 was essentially proven in [13] (see Theorem 6.4 [13]). Proof of Key Theorems 9 in section 11 follows the same scheme.

In Appendix A we study geodesic flows on  $\mathbb{T}^2$  and prove Theorems 4 and 5 about their generic properties.

In Appendix B we derive a normal form for  $H_\epsilon$  at a double resonance.

In Appendix C we study Legendre-Fenchel transform  $\mathcal{LF}_\beta(\lambda h)$ ,  $\lambda > 0$  of an integer homology class  $h \in H_1(\mathbb{T}^2, \mathbb{Z})$  and justify figures 13 and 14 about the shape of channels of cohomologies.

In Appendix D we connect channels of cohomologies from single into double resonances.



## 7 Normally hyperbolic invariant cylinders through the transition zone into double resonances

The main result of this section is Theorem 15. Along with Theorem 4.1 [13] this theorem implies Key Theorem 1 about existence of normally hyperbolic weakly invariant cylinders. Theorem 4.1 [13] applies  $O(\varepsilon^a)$ -away from a (strong) double resonance with  $a \leq 1/4$ . However, we need existence of such cylinders  $O(\sqrt{\varepsilon})$ -away from a double resonance, which is done in Theorem 15. Proof of Theorem 15 follows the proof of Theorem 4.1 [13] with the following modification.

Proofs of both Theorem 4.1 [13] and Theorem 15 consists of two steps: find a normal form  $N_\varepsilon = H_\varepsilon \circ \Phi_\varepsilon$  (Corollary 3.2 [13] and Theorem 17 respectively) and construct an isolating block for  $N_\varepsilon$  (see sections 4 in [13] and section 7.3 respectively). The only difference of the two is that in the normal form theorem we show that

$$N_\varepsilon = H_\varepsilon \circ \Phi_\varepsilon = H_0(\cdot) + \varepsilon Z(\cdot) + \varepsilon R(\cdot, t),$$

where  $\|R\| \leq \delta$  for some small predetermined  $\delta$  and two different norms. In [13] we use the standard  $C^2$ -norm, while in section 7.3 we define a skew-symmetric  $C^2$ -norm (see rescaling (16)). It might well happen that the  $C^2$ -norm of  $\|R\|$  blows up in a  $O(\varepsilon^a)$ -neighborhood of a double resonance for  $a > 1/4$ . This is due to sensitive dependence of action variables. The idea of this rescaling is to stretch action variables by  $1/\sqrt{\varepsilon}$  making partial derivatives of  $R$  in actions less sensitive. It turns out it does not affect applicability of the isolating block arguments as shown in section 7.3.

We recall some notations introduced in section 2.1. Fix  $\vec{k} = (\vec{k}_1, k_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$  and a resonant line  $\Gamma = \Gamma_k = \{p \in B^2 : \vec{k}_1 \cdot \partial_p H_0 + k_0 = 0\} \subset B^2$ . We pick a complementary resonance  $\vec{k}'$ , let  $\theta^s = \vec{k} \cdot (\theta, t)$  and  $\theta^f = \vec{k}' \cdot (\theta, t)$ . We complete it to a  $L : (\theta, p, t, E) \longrightarrow (\theta^s, \theta^f, p^s, p^f, t, E')$ . The averaged perturbation is given by

$$Z(\theta^s, p) = \int \int H_1 \circ L^{-1}(\theta^s, p^s, \theta^f, p^f, t) d\theta^f dt.$$

For a perturbation  $H_1 \in \mathcal{U}_{SR}^\lambda$ , i.e. it satisfies conditions [G0]-[G2] with the parameter  $\lambda > 0$ , we determine a small  $\delta = \delta(\lambda, r) > 0$  and an integer  $K = K(\delta, r, \vec{k})$ , and the set of strong double resonances is defined by

$$\begin{aligned} \Sigma_K &= \{p \in \Gamma \cap B : \exists \vec{k}' = (\vec{k}'_1, k'_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}, \\ &\quad \vec{k}' \not\parallel \vec{k}, \quad |\vec{k}'|, |k'_0| \leq K, \quad \vec{k}'_1 \cdot \partial_p H_0 + k'_0 = 0\}. \end{aligned}$$

In section 2.2 we define the passage segments to be the connected components of the set  $\Gamma \setminus \bigcup_{p_0 \in \Sigma_K} U_{\bar{E}\sqrt{\varepsilon}}(p_0)$ . Roughly speaking, Key Theorem 1 asserts that the following hold for each passage segment.

- On a neighborhood of each passage segment there exists a convenient normal form for the original Hamiltonian  $H_\varepsilon$ .
- Using this normal form one can establish existence of a (weakly) normally hyperbolic invariant cylinder  $\mathcal{C} = \mathcal{C}_{\vec{k}}$  “over” each passage segment.<sup>14</sup> This cylinder is *crumpled* in the sense that it is a graph  $\{(\Theta^s, P^s)(\theta^f, p^f, t) : (\theta^f, p^f, t) \in \mathbb{T} \times [a_j, a_{j+1}] \times \mathbb{T}\}$  and  $\left\| \frac{\partial \Theta^s}{\partial p^f} \right\| \lesssim \varepsilon^{-1/2}$  (see also Figure 7). Asymptotically in  $\varepsilon$ , the maximum of  $\left\| \frac{\partial \Theta^s}{\partial p^f} \right\|$  can be  $\approx \varepsilon^{-1/2}$  near fixed order double resonances (see Remark 7.1).

In [13], the above statements are proved for connected components of the set  $\Gamma \setminus U_{3\varepsilon^{1/6}}(\Sigma_K)$ . We will now focus on proving the same on the set

$$\Gamma \cap U_{5\varepsilon^{1/6}}(\Sigma_K) \setminus U_{\bar{E}\sqrt{\varepsilon}}(\Sigma_K).$$

## 7.1 Normally hyperbolic invariant manifolds going into double resonances

We fix a double resonance

$$p_0 \in \Gamma_{\vec{k}} \cap \Gamma_{\vec{k}'}, \quad \vec{k}' = (k'_1, k'_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}, \vec{k}' \nparallel \vec{k}, |\vec{k}'| \leq K$$

and consider the resonant segment  $\sqrt{\varepsilon}$ -close to  $p_0$ .

Denote by  $p_*^s(p^f) \in \mathbb{R}$  the solution of the equation  $\partial_{p^s} H_0(p_*^s(p^f), p^f) = 0$ . Also denote  $p_*(p^f) := (p_*^s(p^f), p^f)$ . Without loss of generality, we assume  $p^f = 0$  at  $p_0$ . For  $M \gg 1$ , we consider the segment  $p^f \in [M\sqrt{\varepsilon}, 5\varepsilon^{1/6}]$ , which overlaps with the segment  $\Sigma \setminus U_{3\varepsilon^{1/6}}(p_0)$  covered in [13]. We consider the neighborhood

$$\{(\theta, p^f, p^s, t) : p^f \in [M\sqrt{\varepsilon}, 5\varepsilon^{1/6}], \|p_s - p_*^s(p^f)\| \leq \varepsilon\}. \quad (14)$$

which we sometimes refer to the region of interest.

**Theorem 15.** *There exist a small  $\varepsilon_0 > 0$  and a large  $M > 1$  such that for any  $0 < \varepsilon \leq \varepsilon_0 \lambda^{7/2}$  and  $0 \leq \delta \leq \sqrt{\varepsilon_0} \lambda^2$  there exists a  $C^1$  map*

$$(\Theta^s, P^s)(\theta^f, p^f, t) : \mathbb{T} \times [M\sqrt{\varepsilon}, 5\varepsilon^{1/6}] \times \mathbb{T} \longrightarrow \mathbb{T} \times \mathbb{R}$$

---

<sup>14</sup>In this notation we drop dependence of indices  $j$  (index of a resonance) and  $i$  (index of a passage segment in  $\Gamma_j$ ).

such that the cylinder

$$\mathcal{C} = \{(\theta^s, p^s) = (\Theta^s, P^s)(\theta^f, p^f, t); \quad p^f \in [M\sqrt{\epsilon}, 5\epsilon^{1/6}], \quad (\theta^f, t) \in \mathbb{T} \times \mathbb{T}\}$$

is weakly invariant with respect to  $N_\epsilon$  in the sense that the Hamiltonian vector field is tangent to  $\mathcal{C}$ <sup>15</sup>. The cylinder  $\mathcal{C}$  is contained in the set

$$V := \{(\theta, p, t); p^f \in [M\sqrt{\epsilon}, \epsilon^{1/6}], \\ \|\theta^s - \theta_*^s(p^f)\| \leq O(\epsilon_0^{1/4} \lambda), \quad \|p^s - p_*^s(p^f)\| \leq O(\epsilon_0^{1/4} \lambda^{5/4} \epsilon^{1/2})\},$$

and it contains all the full orbits of  $N_\epsilon$  contained in  $V$ . We have the estimates

$$\begin{aligned} \|\Theta^s(\theta^f, p^f, t) - \theta_*^s(p^f)\| &\leq O(\lambda^{-1} \delta + \lambda^{-3/4} \sqrt{\epsilon}), \\ \|P^s(\theta^f, p^f, t) - p_*^s(p^f)\| &\leq \sqrt{\epsilon} O(\lambda^{-3/4} \delta + \lambda^{-1/2} \sqrt{\epsilon}), \\ \left\| \frac{\partial \Theta^s}{\partial p^f} \right\| &\leq O\left(\frac{\epsilon_0^{1/4} \lambda^{-1/4}}{\sqrt{\epsilon}}\right), \quad \left\| \frac{\partial \Theta^s}{\partial(\theta^f, t)} \right\| \leq O\left(\epsilon_0^{1/4} \lambda^{-1/4}\right). \end{aligned}$$

In notation of section 3.2 we set  $\bar{E} = 2M$ .

**Remark 7.1.** Fix a double resonance  $p_0 \in \Gamma \cap \Gamma_{\vec{k}'} \neq \emptyset$ , e.g.  $|\vec{k}'| \in [K, 2K]$ . We have that  $O(\epsilon)$ -close to  $p_0$ ,

$$\max \left\| \frac{\partial \Theta^s}{\partial p^f} \right\| \gtrsim \frac{1}{\sqrt{\epsilon}},$$

where  $\gtrsim$  means that there is a constant  $c$  depending on  $\lambda, \delta, r, H_0, H_1, \Gamma, \vec{k}'$ , but not on  $\epsilon$ .

In the region  $\bar{E}\sqrt{\epsilon}$ -close to  $p_0$ , the double resonance normal form applies. Using results of section B.2, the dynamics of  $H_\epsilon$  is well approximated by dynamics of the corresponding mechanical system  $H^s = K - U$ , after rescaling the action component  $p$  by a factor  $1/\sqrt{\epsilon}$ . By Key Theorem 2 the Hamiltonian  $H_\epsilon$  in  $In(p_0)$  has a normally hyperbolic weakly invariant cylinder  $\mathcal{M}_{h,\epsilon}^{[M,3M]}$  which is a small perturbation of the cylinder  $\mathcal{M}_h^{[M,3M]}$  formed by the union of minimal geodesics  $\gamma_h^E$ 's with  $E \in [M, 3M]$  (see Section 3.2). Generically there is a nontrivial dependence of  $\gamma_h^E$  on  $E$ . Rescaling back into original action variables leads to  $\left\| \frac{\partial \Theta^s}{\partial p^f} \right\| \approx \frac{1}{\sqrt{\epsilon}}$ .

---

<sup>15</sup>As before in notation  $\mathcal{C}$  we drop dependence of indices  $j$  (index of a resonance) and  $i$  (index of a passage segment in  $\Gamma_j$ ).

The rest of the proof is organized as follows. In Section 7.2 we find a proper normal form  $H_\varepsilon \circ \Phi_{TR}$  and get estimates and properties of  $\Phi_{TR}$ . It is done in two steps:

- determine a good normal form for an autonomous Hamiltonian near a double resonance,
- apply this result to our time-periodic case.

In Section 7.3 we construct an isolating block for the normal form system and apply the results of section B, [13] to finish the proof.

## 7.2 A Normal form in the transition zone

### 7.2.1 Autonomous case and slow-fast coordinates

We first state a result for autonomous systems. The time periodic version will come as a corollary. We are interested in a normal form for a Hamiltonian  $\varepsilon^{1/6}$ -near a strong double resonance, but  $M\sqrt{\varepsilon}$ -away from it along one of resonant directions.

Consider the Hamiltonian  $H_\varepsilon(\varphi, J) = H_0(J) + \varepsilon H_1(\varphi, J)$ , where  $(\varphi, J) \in \mathbb{T}^d \times \mathbb{R}^d$  (later, we will take  $d = n + 1$ ). Let  $B = \{|J| \leq 1\}$  be the unit ball in  $\mathbb{R}^d$ . Given any integer vector  $k \in \mathbb{Z}^d \setminus \{0\}$ , let  $[k] = \max\{|k_i|\}$ . To avoid zero denominators in some calculations, we make the unusual convention that  $[(0, \dots, 0)] = 1$ .

Fix a regular energy surface and two linearly independent resonances  $\Gamma_{\vec{k}}$  and  $\Gamma_{\vec{k}'}$ , which intersect at some point  $J_0$ . We order resonances:  $\Gamma_{\vec{k}}$  is the first and  $\Gamma_{\vec{k}'}$  is the second. In the local coordinates near  $J_0$  and notations of section 6.2 we have

$$\theta^s = (\theta^{ss}, \theta^{sf}) \in \mathbb{T}^s \times \mathbb{T}^f \cong \mathbb{T}^2 \times \mathbb{T}^{d-2}$$

with  $\theta^s = (\theta^{ss}, \theta^{sf})$

$$\theta^{ss} = \vec{k}_1 \cdot \theta + k_0 t \text{ and } \theta^{sf} = \vec{k}'_1 \cdot \theta + k'_0 t.$$

The other variables  $\theta^f$  are defined so that change of coordinates from  $\theta$  to  $(\theta^s, \theta^f)$  is given some matrix  $A \in SL_d(\mathbb{Z})$ . Define a symplectic linear change of coordinates

$$L : \begin{bmatrix} \theta \\ J \end{bmatrix} \longrightarrow \begin{bmatrix} \theta^s \\ \theta^f \\ J^s \\ J^f \end{bmatrix} \longrightarrow \begin{bmatrix} A \theta \\ A^* J \end{bmatrix} \quad \text{with} \quad A^* = (A^{-1})^T.$$

Denote action variables  $J = (J^{ss}, J^{sf}, J^f)$  conjugate to  $\theta = (\theta^{ss}, \theta^{sf}, \theta^f)$  and  $J^s = (J^{ss}, J^{sf})$ . Consider the Hamiltonian in the new coordinates

$$H_\varepsilon(\theta^s, J^s, \theta^f, J^f) = H_0(J) + \varepsilon H_1(\theta^s, J^s, \theta^f, J^f).^{16} \quad (15)$$

---

<sup>16</sup>We somewhat abuse notations by denoting  $H_0$  and  $H_1$  by the same letter. We hope it does not lead to confusions.

Call these coordinates *slow-fast*. Note that they distinguish three time scales: slow, slow-fast, and fast. In these local coordinates  $\Gamma_{\vec{k}} = \{J^{ss} = 0\}$ . Fix  $1 \ll m \ll M$ , where  $m$  depends on norms of  $H_0$  and  $H_1$ , while  $M$  will be specified later. We are interested in a dynamics in a  $m\sqrt{\varepsilon}$ -neighborhood of  $\Gamma_{\vec{k}}$  with  $J^{sf} \in [M\sqrt{\varepsilon}, \varepsilon^{1/6}]$ . Due to the implicit function theorem and convexity of  $H_0$  from  $H_0(J_0) = H_0(0, J_*^{sf}, J^f)$  one can define a function  $J^f = J^f(J_*^{sf})$  satisfying this condition. Define in local coordinates

$$\mathcal{D}(m, M, J_*^{sf}, \varepsilon) = \{J : |J - (0, J_*^{sf}, J^f(J_*^{sf}))| \leq m\sqrt{\varepsilon}\}.$$

We are looking for normalizing coordinate changes to average out slow-fast and slow motions. It turns out that these changes depend on *slow-fast and fast action components in much more sensitive than on slow actions*. To compensate this consider a linear change of coordinates:

$$\begin{aligned} L_{\varepsilon, J_*^{sf}} : (\theta^{ss}, \theta^{sf}, \theta^f, \tilde{J}^{ss}, \tilde{J}^{sf}, \tilde{J}^f) &\longrightarrow \\ (\theta^{ss}, \theta^{sf}, \theta^f, J^{ss}, J^{sf}, J^f) &= (\theta^{ss}, \theta^{sf}, \theta^f, \tilde{J}^{ss}, J_*^{sf} + \sqrt{\varepsilon}\tilde{J}^{sf}, \sqrt{\varepsilon}\tilde{J}^f). \end{aligned} \quad (16)$$

**Theorem 16.** *Fix parameters  $r \geq d+4$ ,  $d > 1$  and  $\delta \in (0, 1)$ . There exists a constant  $c = c_d > 0$ , which depends only on  $d$ , such that the following holds.*

*Let  $H_0(J)$  be  $C^4$  and  $H_1(\theta, J)$  be  $C^r$  with  $\|H_1\|_{C^r} = 1$ . Then for sufficiently small  $\varepsilon > 0$  and  $K \geq c\delta^{\frac{-1}{r-d-3}}$  there exists a  $C^2$  symplectic diffeomorphism  $\Phi$  such that, in the new coordinates, the Hamiltonian  $H_\varepsilon = H_0 + \varepsilon H_1$  takes the form*

$$H_\varepsilon \circ \Phi = H_0 + \varepsilon R_1(\theta^{ss}, J) + \varepsilon R_2(\theta, J)$$

*with  $R_1 = \sum_{k \in \mathbb{Z}^d, |k| \leq K, (k^{sf}, k^f)=0} h_k(J) e^{2\pi i(k \cdot \theta)}$ , here  $h_k(J)$  is the  $k^{\text{th}}$  coefficient for the Fourier expansion of  $H_1$ .*

*For a sufficiently large  $M = M(\delta, H_0, H_1)$  and each  $J_*^{sf} \in [M\sqrt{\varepsilon}, \varepsilon^{1/6}]$  we have*

$$\begin{aligned} H_\varepsilon \circ \Phi \circ L_{\varepsilon, J_*^{sf}} &= \\ &= H_0 \circ L_{\varepsilon, J_*^{sf}} + \varepsilon R_1 \circ L_{\varepsilon, J_*^{sf}}(\theta^{ss}, \tilde{J}) + \varepsilon R_2 \circ L_{\varepsilon, J_*^{sf}}(\theta, \tilde{J}) \\ &=: H_0 + \varepsilon \tilde{R}_1(\theta^{ss}, \tilde{J}) + \varepsilon \tilde{R}_2(\theta, \tilde{J}) \\ \|\tilde{R}_2\|_{C^2} &\leq \delta \quad \text{on} \quad \mathbb{T} \times L_{\varepsilon, J_*^{sf}}^{-1} \mathcal{D}(m, M, J_*^{sf}, \varepsilon) \times \mathbb{T}^{m-1} \\ \|\Phi \circ L_{\varepsilon, J_*^{sf}} - Id\|_{C^l} &\leq \delta \sqrt{\varepsilon} \quad \text{for} \quad l = 0, 1, 2. \end{aligned}$$

Note that  $\Phi \circ L_{\varepsilon, J_*^{sf}}$  should not be viewed as a change of coordinates. It is a rather a convenient way to hide blow up of partial derivatives of  $\Phi$  with respect to slow-fast and fast action variables.

To prove Theorem 16 we need the following basic estimates about the Fourier series of a function  $g(\varphi, J)$ . Given a multi-index  $\alpha = (\alpha_1, \dots, \alpha_d)$ , we denote  $|\alpha| = \alpha_1 + \dots + \alpha_d$ . Denote also  $\kappa = \kappa_d = \sum_{\mathbb{Z}^d} [k]^{-d-1}$ . To avoid cumbersome notations we will denote by  $c$  various constants independent of all parameters of the problem, but  $d$ .

**Lemma 7.1.** (see e.g. [13], lemma 3.1) For  $g(\varphi, J) \in C^r(\mathbb{T}^d \times B)$ , we have

1. If  $l \leq r$ , we have  $\|g_k(J)e^{2\pi i(k \cdot \varphi)}\|_{C^l} \leq [k]^{l-r} \|g\|_{C^r}$ .
2. Let  $g_k(J)$  be a series of functions satisfying  $\|\partial_{J^\alpha} g_k\|_{C^0} \leq M[k]^{-|\alpha|-d-1}$  holds for each multi-index  $\alpha$  with  $|\alpha| \leq l$ , for some  $M > 0$ . Then, we have the following bound  $\|\sum_{k \in \mathbb{Z}^d} g_k(J)e^{2\pi i(k \cdot \varphi)}\|_{C^l} \leq c\kappa M$ .
3. Let  $\Pi_K^+ g = \sum_{|k| > K} g_k(J)e^{2\pi i(k \cdot \varphi)}$ . Then for  $l \leq r - d - 1$ , we have  $\|\Pi_K^+ g\|_{C^l} \leq \kappa K^{d-r+l+1} \|g\|_{C^r}$ .

*Proof of Theorem 16.* To simplify notations let

$$I = (I_1, I_2, I_f) = (J^{ss}, J^{sf}, J^f), \quad J^* = (0, J_*^{sf}, J^f(J_*^{sf})),$$

$$\varphi = (\varphi_1, \varphi_2, \varphi_f) = (\theta^{ss}, \theta^{sf}, \theta^f).$$

Let  $G(\varphi, I)$  be the function that solves the cohomological equation

$$\{H_0, G\} + H_1 = R_1 + R_+,$$

where  $R_+ = \Pi_K^+ H_1$ . We have the following explicit formula for  $G$ :

$$G(\varphi, I) = \sum_{|k| \leq K, (k^{sf}, k^f) \neq 0} \frac{h_k(I)}{k \cdot \partial_I H_0} e^{2\pi i(k \cdot \varphi)}.$$

Let  $\Phi^t$  be the Hamiltonian flow generated by  $\epsilon G$ . Setting  $F_t = R_1 + R_+ + t(H_1 - R_1 - R_+)$ , we have the standard computation

$$\begin{aligned} \partial_t((H_0 + \epsilon F_t) \circ \Phi^t) &= \epsilon \partial_t F_t \circ \Phi^t + \epsilon \{H_0 + \epsilon F_t, G\} \circ \Phi^t \\ &= \epsilon (\partial_t F_t + \{H_0, G\}) \circ \Phi^t + \epsilon^2 \{F_t, G\} \circ \Phi^t \\ &= \epsilon^2 \{F_t, G\} \circ \Phi^t, \end{aligned}$$

Fix  $I_2 = J_*^{sf} \in [M\sqrt{\epsilon}, 5\epsilon^{1/6}]$ . Notice that difference with calculations of the proof of Theorem 3.2 [13] is two fold:

— we do not use  $\rho$ -mollifiers and

— consider skew-symmetric norms of the remainder with respect to rescaled variables. Adapting notations we have  $L_{\varepsilon, I} : (\varphi, \tilde{I}) \longrightarrow (\varphi, I)$  with  $I_2 = I_2^* + \sqrt{\varepsilon} \tilde{I}_2$ ,  $I^f = I_*^f + \sqrt{\varepsilon} \tilde{I}^f$ . The key feature is that after rescaling derivatives with respect to slow-fast and fast action variables have an additional  $\sqrt{\varepsilon}$ -factor.

Let us estimate the  $C^2$  norm of the function  $R_2 := R_+ + \varepsilon \int_0^1 \{F_t, G\} \circ \Phi^t dt$ . It follows from Lemma 7.1 that

$$\|R_+\|_{C^2} \leq \kappa K^{-r+5} \|H_1\|_{C^r} \leq \frac{1}{2} \delta.$$

We now focus on the term  $\int_0^1 \{F_t, G\} \circ \Phi^t dt$ . To estimate the norm of  $F_t$ , it is convenient to write  $F_t = \tilde{F}_t + (1-t)R_1$ , where  $\tilde{F}_t = (1-t)R_+ + tH_1$ . Notice that the coefficients of the Fourier expansion of  $\tilde{F}_t$  is simply a constant times that of  $H_1$ , Lemma 7.1 then implies that

$$\|\tilde{F}_t\|_{C^3} \leq \sum_{k \in \mathbb{Z}^3} [k]^{3-r} \|H_1\|_{C^r} = \kappa \|H_1\|_{C^r}$$

provided that  $r \geq m+4$ , where as before  $\kappa = \sum_{\mathbb{Z}^3} [k]^{-4}$ . The same estimate applies to  $R_1$ . Therefore,

$$\|\tilde{F}_t\|_{C^3} \leq \|R_t\|_{C^3} + \|\tilde{F}_t\|_{C^3} \leq 2\kappa \|H_1\|_{C^r}.$$

For  $l \in \{0, 1, 2, 3\}$  in rescaled variables using  $J^{sf} > C\sqrt{\varepsilon}$  we have the following estimates:

$$\|(k \cdot \partial_{\tilde{J}} H_0)^{-1}\|_{C^l} \leq \frac{m \|H_0\|_{C^4}^{l+1}}{M \sqrt{\varepsilon}}.$$

To estimate norms of  $G$  we use the following estimates on the derivative of composition of functions: For  $f : \mathbb{R}^d \longrightarrow \mathbb{R}$  and  $g : \mathbb{R}^d \longrightarrow \mathbb{R}^d$  we have

$$\|f \circ g\|_{C^l} \leq c_{d,l} \|f\|_{C^l} (1 + \|g\|_{C^l}^l).$$

For each multi-index  $|\alpha| \leq 3$  and  $(k^{sf}, k^f) \neq 0$ , we have that

$$\begin{aligned} \|\partial_{\tilde{I}^\alpha} (h_k(\tilde{I})(k \cdot \partial_{\tilde{J}} H_0)^{-1})\|_{C^0} &\leq \sum_{\alpha_1 + \alpha_2 = \alpha} \|h_k\|_{C^{|\alpha_1|}} \|(k \cdot \partial_{\tilde{J}} H_0)^{-1}\|_{C^{|\alpha_2|}} \\ &\leq \sum_{\alpha_1 + \alpha_2 = \alpha} [k]^{-r+|\alpha_1|} \|H_1\|_{C^r} \cdot \frac{m \|H_0\|_{C^4}^{|\alpha_2|}}{M \sqrt{\varepsilon}} \leq m [k]^{-r+|\alpha|} \frac{\|H_0\|_{C^4}^{|\alpha|+1} \|H_1\|_{C^r}}{M \sqrt{\varepsilon}} \end{aligned}$$

This implies that

$$\|G(\varphi, \tilde{I})\|_{C^l} \leq m \frac{\|H_1\|_{C^r} \|H_0\|_{C^4}^{l+1}}{M \sqrt{\varepsilon}} \quad \text{for } l = 0, 1, 2, 3.$$

We now apply our estimates to

$$\|\{F_t, G\}\|_{C^2} \leq \sum_{|\alpha_1 + \alpha_2| \leq 3} \|F_t\|_{C^{|\alpha_1|}} \|G\|_{C^{|\alpha_2|}} \leq \frac{m \|H_1\|_{C^r}^2 \|H_0\|_{C^4}^4}{M \sqrt{\epsilon}}.$$

Notice that the rescaled flow  $\Phi^t \circ L_{\epsilon, J_*^{sf}}$  satisfies

$$\|\epsilon G\|_{C^3} \leq m \epsilon \frac{\|H_1\|_{C^r}^2 \|H_0\|_{C^4}^{l+1}}{M^2 \sqrt{\epsilon}} \ll 1.$$

Choosing  $M$  appropriately for  $l = 0, 1, 2$  we get the following estimate (see *e. g.* [29], Lemma 3.15):

$$\|\Phi^t - Id\|_{C^l} \leq m \epsilon \|G\|_{C^{l+1}} \leq \sqrt{\epsilon} \|H_1\|_{C^r}^2 \|H_0\|_{C^4}^{l+1} \frac{m}{M}.$$

□

### 7.2.2 Time-periodic setting and reduction to $d = 3$

Consider a time-periodic Hamiltonian  $H_\epsilon(\theta, p, t) = H_0(p) + \epsilon H_1(\theta, p, t)$  with  $(\theta, p) \in \mathbb{T}^2 \times \mathbb{R}^2$ ,  $t \in \mathbb{T}$ . We denote by  $p_0$  the intersection of the resonance  $\Gamma_{\vec{k}}$  and  $\Gamma_{\vec{k}'}$ . This means

$$\vec{k}_1 \cdot \partial_p H(p_0) + k_0 = 0, \quad \vec{k}'_1 \cdot \partial_p H(p_0) + k'_0 = 0.$$

We consider the autonomous version of the system

$$H_\epsilon(\theta, p, t, E) = H_0(p) + \epsilon H_1(\theta, p, t) + E.$$

We can rewrite the Hamiltonian in the form

$$H_\epsilon(\theta, p, t, E) = H_0(p) + \epsilon H_1(\vec{k}_1 \cdot \theta + k_0, \vec{k}'_1 \cdot \theta + k'_0, p, t) + E,$$

Denote  $\theta^{ss} = \vec{k}_1 \cdot \theta + k_0$ ,  $\theta^{sf} = \vec{k}'_1 \cdot \theta + k'_0$ ,  $\theta^f = t$ , and  $\theta^s = (\theta^{ss}, \theta^{sf})$ , we further rewrite

$$H_\epsilon(\theta^{ss}, \theta^{sf}, J^{ss}, J^{sf}, t, E) = H_0(p^s) + E + \epsilon \tilde{H}_1(\theta^s, p, t).$$

Note that to make the coordinate change  $(\theta^s, p^s, t, E) \rightarrow (\theta^s, p^s, \theta^f, J^f)$  symplectic, the conjugate coordinates  $J^s = (J^{ss}, J^{sf})$  and  $J^f$  should satisfy

$$\begin{bmatrix} p \\ E \end{bmatrix} = \begin{bmatrix} B^T & 0 \\ k_0, k'_0 & 1 \end{bmatrix} \begin{bmatrix} J^s \\ J^f \end{bmatrix}, \quad \text{where } B = \begin{bmatrix} \vec{k}_1 \\ \vec{k}'_1 \end{bmatrix}.$$

Substituting in we have the Hamiltonian

$$H_\epsilon(\theta^{ss}, \theta^{sf}, J^{ss}, J^{sf}, t) + J^f = (H_0(J^{ss}, J^{sf}) + J^f) + \epsilon \tilde{H}_1(\theta^{ss}, \theta^{sf}, J^{ss}, J^{sf}, t).$$

This Hamiltonian is in the form (15) Applying Theorem 16 get the following



**Theorem 17.** *[Normal Form] Let  $p_0$  be a strong double resonance  $p_0 \in \Gamma_{\vec{k}} \cap \Gamma_{\vec{k}'}$  and let  $H_0(p)$  be a  $C^4$  Hamiltonian written in the local slow-fast coordinates defined above.*

*Then for each  $\delta \in (0, 1)$ ,  $m > 1$ , and  $r \geq 7$ , there exist positive parameters  $K_0, \epsilon_0, M$  such that, for each  $C^r$  Hamiltonian  $H_1$  with  $\|H_1\|_{C^r} \leq 1$  and each  $K_0 \leq K$ ,  $0 \leq \epsilon \leq \epsilon_0$ , there exists a  $C^2$  change of coordinates*

$$\Phi_{TR} : \mathbb{T}^2 \times B \times \mathbb{T} \longrightarrow \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T}$$

*defined in  $\mathcal{D}(m, M, J_*^{sf}, \epsilon)$  for each  $J_*^{sf} \in [M\sqrt{\epsilon}, 5\epsilon^{1/6}]$  and such that composition of  $\Phi$  with a linear rescaling  $L_{\epsilon, J_*^{sf}}$  satisfies*

$$\|\Phi_{TR} \circ L_{\epsilon, J_*^{sf}} - Id\|_{C^l} \leq \delta\sqrt{\epsilon} \quad \text{for each } l = 0, 1, 2$$

*and such that, in the new coordinates, the Hamiltonian  $H_0 + \epsilon H_1$  takes the form*

$$N_\epsilon = H_\epsilon \circ \Phi_{TR} = H_0(p) + \epsilon Z(\theta^{ss}, p) + \epsilon R(\theta, p, t), \quad (17)$$

*where*

$$Z(\theta^{ss}, p) = \sum_{k \in \mathbb{Z}^3, |k| < K, (k^{sf}, k^f) = 0} h_k(p) e^{2\pi i k^{ss} \theta^{ss}}$$

*and*

$$\|R \circ L_{\epsilon, J_*^{sf}}\|_{C^2} \leq \delta \quad \text{on} \quad \mathbb{T} \times L_{\epsilon, J_*^{sf}}^{-1} \mathcal{D}(m, M, J_*^{sf}, \epsilon) \times \mathbb{T}^2.$$

## 7.3 Construction of an isolating block

### 7.3.1 Auxiliary estimates on the vector field.

Consider the equation of motion :

$$\begin{cases} \dot{\theta}^s = \partial_{p^s} H_0 + \epsilon \partial_{p^s} Z + \epsilon \partial_{p^s} R \\ \dot{p}^s = -\epsilon \partial_{\theta^s} Z - \epsilon \partial_{\theta^s} R \\ \dot{\theta}^f = \partial_{p^f} H_0 + \varepsilon \partial_{p^f} Z + \varepsilon \partial_{p^f} R \\ \dot{p}^f = -\varepsilon \partial_{\theta^f} R \\ \dot{t} = 1 \end{cases} \quad (18)$$

It is convenient to treat all variables as those on the lines. Then the system is defined on  $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ . We will show that this system is a perturbation of the model system

$$\dot{\theta}^s = \partial_{p^s} H_0 \quad , \quad \dot{p}^s = -\epsilon \partial_{\theta^s} Z \quad , \quad \dot{\theta}^f = \partial_{p^f} H_0 \quad , \quad \dot{p}^f = 0 \quad , \quad \dot{t} = 1. \quad (19)$$

By construction the graph of the map

$$(\theta^f, p^f, t) \longrightarrow (\theta_*^s(p_f), p_*^s(p_f))$$

on  $\mathbb{T} \times [M\sqrt{\varepsilon}, 5\varepsilon^{1/6}] \times \mathbb{T}$  is invariant for the model flow. For each fixed  $p_f$ , the point  $(\theta_*^s(p_f), p_*^s(p_f))$  is a hyperbolic fixed point of the partial system

$$\dot{\theta}^s = \partial_{p^s} H_0(p^s, p^f) \quad , \quad \dot{p}^s = -\epsilon \partial_{\theta^s} Z(\theta^s, p^s, p^f)$$

where  $p^f$  is seen as a parameter. This hyperbolicity is the key property we will use, through the theory of normally hyperbolic invariant manifolds. We notice that calculations below are similar to those in section 3.3 [13]. *The major difference is that the remainder  $R$  is not necessarily  $C^2$ -small.* Notice that both size of hyperbolicity and size of perturbation are  $\varepsilon$ -dependent so there is a competition and application of this theory is not straightforward. On top of that we have to deal with the problem of non-invariant boundaries. We will however manage to apply the quantitative version exposed in Appendix B, [13].

We perform some changes of coordinates in order to put the system in the framework of Appendix B, [13]. These coordinates appear naturally from the study of the model system as follows. We set

$$b(p^f) := \partial_{p^s p^s}^2 H_0(p_*^s(p^f)) \quad , \quad a(p_f) := -\partial_{\theta^s \theta^s}^2 Z(\theta_*^s(p^f), p_*^s(p^f)).$$

If we fix the variable  $p^f$  and consider the model system in  $(\theta^s, p^s)$ , we observed that this system has a hyperbolic fixed point at  $(\theta_*^s(p^f), p_*^s(p^f))$ . The linearized system at this point is

$$\dot{\theta}^s = b(p^f) p^s \quad , \quad \dot{p}^s = \epsilon a(p_f) \theta^s. \quad (20)$$

To put this system under a simpler form, it is useful to introduce two parameters

$$T(p^f) := (a^{-1}(p^f)b(p^f))^{1/4}, \quad \Lambda(p^f) := T^2(p^f)a(p^f).$$

In the new variables

$$\xi = T^{-1}(p^f)\theta^s + \epsilon^{-1/2}T(p^f)p^s \quad , \quad \eta = T^{-1}(p^f)\theta^s - \epsilon^{-1/2}T(p^f)p^s,$$

the linearized system is reduced to the following block-diagonal form:

$$\dot{\xi} = \epsilon^{1/2}\Lambda(p^f)\xi \quad , \quad \dot{\eta} = -\epsilon^{1/2}\Lambda(p^f)\eta.$$

$$\begin{aligned} x &= T^{-1}(p^f)(\theta^s - \theta_*^s(p^f)) + \epsilon^{-1/2}T(p^f)(p^s - p_*^s(p^f)) \\ y &= T^{-1}(p^f)(\theta^s - \theta_*^s(p^f)) - \epsilon^{-1/2}T(p^f)(p^s - p_*^s(p^f)), \\ I^f &= \epsilon^{-1/2}p^f \quad , \quad \Theta = \gamma\theta^f, \end{aligned} \quad (21)$$

where  $\gamma$  is a parameter which will be taken later equal to  $\delta^{1/2}$ . Note that

$$\begin{aligned}\theta^s &= \theta_*^s(\epsilon^{1/2}I^f) + \frac{1}{2}T(\epsilon^{1/2}I^f)(x+y), \\ p^s &= p_*^s(\epsilon^{1/2}I^f) + \frac{\epsilon^{1/2}}{2}T^{-1}(\epsilon^{1/2}I^f)(x-y).\end{aligned}$$

The next three lemmas are proven in [13].

**Lemma 7.2.** *We have  $\Lambda(p^f) \geq \sqrt{\lambda/D} I$  for each  $p^f \in [M\sqrt{\epsilon}, 5\epsilon^{1/6}]$ .*

**Lemma 7.3.** *On the domain  $\|x\| \leq \rho, \|y\| \leq \rho$ , we have the estimates*

$$\begin{aligned}\|T\| &= O(\lambda^{-1/4}), \quad \|T^{-1}\| = O(1), \quad \|\partial_{p^f}T\| \leq O(\lambda^{-3/2}), \\ \|\partial_{p^f}T^{-3/2}\| &\leq O(\lambda^{-3/4}), \quad \|\partial_{p^f}\theta_*^s\| \leq O(\lambda^{-1}), \\ \|p_*^s\|_{C^2} &= O(1), \quad \|\theta^s - \theta_*^s\| \leq O(\lambda^{-1/4}\rho), \quad \|p^s - p_*^s\| \leq O(\epsilon^{1/2}\rho).\end{aligned}$$

**Lemma 7.4.** *The equations of motion in the new coordinates take the form*

$$\begin{aligned}\dot{x} &= -\sqrt{\epsilon}\Lambda(\sqrt{\epsilon}I^f)x + \sqrt{\epsilon}O(\lambda^{-1/4}\delta + \lambda^{-3/4}\rho^2) + O(\epsilon) \\ \dot{y} &= \sqrt{\epsilon}\Lambda(\sqrt{\epsilon}I^f)y + \sqrt{\epsilon}O(\lambda^{-1/4}\delta + \lambda^{-3/4}\rho^2) + O(\epsilon) \\ \dot{I}^f &= O(\sqrt{\epsilon}\delta),\end{aligned}$$

where  $\rho = \max(\|x\|, \|y\|)$  is assumed to satisfy  $\rho \leq \lambda$ . The expression for  $\dot{\Theta}$  is not useful here.

**Lemma 7.5.** *In the new coordinate system  $(x, y, \Theta, I, t)$ , the linearized system is given by the matrix*

$$L = \begin{bmatrix} \sqrt{\epsilon}\Lambda & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\epsilon}\Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + O(\lambda^{-3/4}\rho\sqrt{\epsilon} + \delta\lambda^{-1/4}\sqrt{\epsilon} + \lambda^{-5/4}\epsilon + \gamma\sqrt{\epsilon}),$$

where  $\rho = \max(\|x\|, \|y\|)$ .

*Proof of Lemma 7.5.* Most of the estimates below are based on Lemma 7.3.

In the original coordinates, the matrix of the linearized system is:

$$\tilde{L} = \begin{bmatrix} O(\epsilon) & \partial_{p^s p^s}^2 H_0 + O(\epsilon) & 0 & \partial_{p^f p^s}^2 H_0 + O(\epsilon) & 0 \\ -\epsilon \partial_{\theta^s \theta^s}^2 Z & O(\epsilon) & 0 & O(\epsilon) & 0 \\ O(\epsilon) & \partial_{p^f p^s}^2 H_0 + O(\epsilon) & O(\delta\epsilon) & \partial_{p^f p^f}^2 H_0 + O(\epsilon) & O(\delta\epsilon) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\epsilon \cdot \begin{bmatrix} 0 & 0 & 0 & \partial_{p^f p^s}^2 R & 0 \\ 0 & 0 & 0 & -\partial_{p^f \theta^s}^2 R & 0 \\ \partial_{p^f \theta^s}^2 R & \partial_{p^f p^s}^2 R & \partial_{p^f \theta^f}^2 R & \partial_{p^f p^f}^2 R & \partial_{p^f t}^2 R \\ 0 & 0 & 0 & -\partial_{p^f \theta^f}^2 R & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In our notations for the first part of the matrix we have

$$\begin{bmatrix} O(\epsilon) & B + O(\epsilon + \sqrt{\epsilon}\rho) & 0 & \partial_{p^f p^s}^2 H_0 + O(\epsilon) & 0 \\ -\epsilon A + O(\epsilon \lambda^{-1/4} \rho) & O(\epsilon) & 0 & O(\epsilon) & 0 \\ O(\epsilon) & O(1) & O(\delta\epsilon) & O(1) & O(\delta\epsilon) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the new coordinates, the matrix is the product

$$L = \left[ \frac{\partial(x, y, \Theta, I, t)}{\partial(\theta^s, p^s, \theta^f, p^f, t)} \right] \cdot \tilde{L} \cdot \left[ \frac{\partial(\theta^s, p^s, \theta^f, p^f, t)}{\partial(x, y, \Theta, I, t)} \right].$$

We have

$$\left[ \frac{\partial(\theta^s, p^s, \theta^f, p^f, t)}{\partial(x, y, \Theta, I, t)} \right] = \begin{bmatrix} T/2 & T/2 & 0 & O(\sqrt{\epsilon}\lambda^{-1}) & 0 \\ \sqrt{\epsilon}T^{-1}/2 & -\sqrt{\epsilon}T^{-1}/2 & 0 & \sqrt{\epsilon}\partial_{p^f p^s}^2 + O(\epsilon\lambda^{-3/4}\rho) & 0 \\ 0 & 0 & \gamma^{-1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\epsilon} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\begin{aligned} \tilde{L} \left[ \frac{\partial(\theta^s, p^s, \theta^f, p^f, t)}{\partial(x, y, \Theta, I, t)} \right] &= \frac{\sqrt{\varepsilon}}{2} \times \\ &\times \begin{bmatrix} BT^{-1} + O(\sqrt{\varepsilon}\lambda^{-1/4}) & -BT^{-1} + O(\sqrt{\varepsilon}\lambda^{-1/4}) & 0 & O(\sqrt{\varepsilon}\lambda^{-3/4}\rho + \varepsilon\lambda^{-1}) & 0 \\ AT + O(\sqrt{\varepsilon}\lambda^{-1/2}\rho) & \sqrt{\varepsilon}AT + O(\sqrt{\varepsilon}\lambda^{-1/2}\rho) & 0 & \varepsilon O(\lambda^{-5/4}\rho + \lambda^{-1}) & 0 \\ O(1) & O(1) & O(\gamma^{-1}\delta\sqrt{\varepsilon}) & O(1) & O(\delta\sqrt{\varepsilon}) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &+ \varepsilon \cdot \begin{bmatrix} 0 & 0 & 0 & \sqrt{\varepsilon}\partial_{p^f p^s} R & 0 \\ 0 & 0 & 0 & -\sqrt{\varepsilon}\partial_{p^f \theta^s} R & 0 \\ D_+ & D_- & \gamma^{-1}\partial_{p^f \theta^f} R & D & \partial_{p^f t} R \\ 0 & 0 & 0 & -\sqrt{\varepsilon}\partial_{p^f \theta^f} R & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} D_{\pm} &= \partial_{p^f \theta^s} R \cdot T \pm \sqrt{\varepsilon} \partial_{p^f p^s} R \cdot T^{-1}, \\ D &= \sqrt{\varepsilon} [\partial_{p^f \theta^s} R O(\lambda^{-1}) + \partial_{p^f p^s} R (\partial_{p^f p^s}^s + \sqrt{\varepsilon} O(\lambda^{-3/4}\rho)) + \partial_{p^f p^f} R]. \end{aligned} \quad (22)$$

This expression is the result of a tedious, but obvious, computation. Let us just detail the computation of the coefficient on the first line, fourth row which contain an important cancelation:

$$\begin{aligned} &\sqrt{\varepsilon}\partial_{p^s p^s}^2 H_0 \partial_{p^f p^s}(p^f) + \sqrt{\varepsilon}\partial_{p^f p^s}^2 H_0 + O(\varepsilon\lambda^{-3/4}\rho + \varepsilon^{3/2}\lambda^{-1}) \\ &= \sqrt{\varepsilon}\partial_{p^f} (\partial_{p^s} H_0(p^s(p^f), p^f)) + O(\varepsilon\lambda^{-3/4}\rho + \varepsilon^{3/2}\lambda^{-1}) = O(\varepsilon\lambda^{-3/4}\rho + \varepsilon^{3/2}\lambda^{-1}). \end{aligned}$$

We now write

$$\left[ \frac{\partial(x, y, \Theta, I, t)}{\partial(\theta^s, p^s, \theta^f, p^f, t)} \right] = \begin{bmatrix} T^{-1} & \varepsilon^{-1/2}T & 0 & O(\varepsilon^{-1/2}\lambda^{-1/4}) & 0 \\ T^{-1} & -\varepsilon^{-1/2}T & 0 & O(\varepsilon^{-1/2}\lambda^{-1/4}) & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and compute that

$$L = \begin{bmatrix} \sqrt{\epsilon}\Lambda + O(\sqrt{\epsilon}\lambda^{-3/4}\rho) & O(\sqrt{\epsilon}\lambda^{-3/4}\rho) & 0 & O(\epsilon\lambda^{-5/4}) & 0 \\ O(\epsilon\lambda^{-3/4}\rho) & -\sqrt{\epsilon}\Lambda + O(\sqrt{\epsilon}\lambda^{-3/4}\rho) & 0 & O(\epsilon\lambda^{-5/4}) & 0 \\ O(\gamma\sqrt{\epsilon}) & O(\gamma\sqrt{\epsilon}) & O(\delta\epsilon) & O(\gamma\sqrt{\epsilon}) & O(\delta\gamma\epsilon) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ + \epsilon \cdot \begin{bmatrix} 0 & 0 & 0 & B_+ & 0 \\ 0 & 0 & 0 & B_- & 0 \\ \gamma D_+ & \gamma D_- & \partial_{p^f\theta^f} R & \gamma D & \gamma \partial_{p^f t} R \\ 0 & 0 & 0 & -\partial_{p^f\theta^f} R & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$B_{\pm} = \sqrt{\epsilon} T^{-1} \partial_{p^f p^s} R \pm T \partial_{p^f \theta^s} R + O(\lambda^{-1/4}) \partial_{p^f \theta^f} R.$$

Now we substitute our knowledge of derivatives. By Theorem 17 we have

$$|\partial_{p^f \theta^s}^2 R|, |\partial_{p^f p^s}^2 R| \leq \frac{\delta}{\sqrt{\epsilon}}, \quad |\partial_{p^f p^f}^2 R| \leq \frac{\delta}{\epsilon}.$$

Substituting this into  $B$ 's and  $D$ 's we have

$$\begin{aligned} \epsilon |D| &\leq \delta O(\epsilon\lambda^{-1} + \sqrt{\epsilon}\lambda^{-3/4}\rho + \sqrt{\epsilon}). \\ \epsilon |D_{\pm}| &\leq \delta O(\epsilon + \sqrt{\epsilon}). \\ \epsilon |B_{\pm}| &\leq \delta O(\epsilon + \sqrt{\epsilon} + \lambda^{-1/4}\sqrt{\epsilon}). \end{aligned}$$

Conclude the second part of the linearization  $L$  has entries bounded by  $O(\delta \sqrt{\epsilon} \lambda^{-1/4})$ . This completes the proof.  $\square$

### 7.3.2 Constructing the isolation block

In order to prove the existence of a normally hyperbolic invariant strip (for the lifted system), we apply Proposition B.3 [13] to the system in coordinates  $(x, y, \Theta, I, t)$ . More precisely, with the notations of Appendix B [13], we set:

$$u = x, s = y, c_1 = (\Theta, t), c_2 = I, \Omega = \mathbb{R}^2 \times \Omega^{c_2} = \mathbb{R}^2 \times [M\sqrt{\epsilon}, \epsilon^{1/6}].$$

We fix  $\gamma = \sqrt{\delta}$  and  $\alpha = \sqrt{\epsilon\lambda/4D}$ , recall that  $\sqrt{\epsilon}\Lambda \geq 2\alpha I$ , by Lemma 7.2. We take  $\sigma = \lambda\epsilon^{-1/2}/2$ , so that

$$\Omega_{\sigma} = \mathbb{R}^2 \times [M\sqrt{\epsilon}, \epsilon^{1/6}].$$

We assume that  $\epsilon$  satisfies  $0 < \epsilon < \epsilon_0 \lambda^{7/2}$  and  $0 \leq \delta < \sqrt{\epsilon_0} \lambda^2$ . We can apply Proposition B.3 [13] with  $B^u = \{u : \|u\| \leq \rho\}$  and  $B^s = \{s : \|s\| \leq \rho\}$  provided

$$\epsilon_0^{-1/4}(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\epsilon}) \leq \rho \leq 2\epsilon_0^{1/4}\lambda^{5/4}. \quad (23)$$

It is easy to check under our assumptions on the parameters that such values of  $\rho$  exist. These estimates along with Lemma 7.3 imply that

$$\|\theta^s - \theta_*^s(p^f)\| \leq O(\epsilon_0^{1/4}\lambda) \quad , \quad \|p^s - p_*^s(p^f)\| \leq O(\epsilon_0^{1/4}\lambda^{5/4}\epsilon^{1/2}).$$

Provided that the cylinder  $\mathcal{C}$  exists, this gives the first set of estimates in Theorem 15.

Let us check the isolating block condition. By Lemma 7.4, we have

$$\dot{x} \cdot x \geq 2\alpha\|x\|^2 - \|x\| O(\epsilon^{1/2}\lambda^{-1/4}\delta + \epsilon^{1/2}\lambda^{-3/4}\rho^2 + \epsilon)$$

if  $x \in B^u, y \in B^s$ . If in addition  $\|x\| = \rho$ , then from the lower bound on  $\rho$  we have

$$\lambda^{-3/4}\delta \leq \epsilon_0^{1/4}\|x\| \quad , \quad \lambda^{-5/4}\rho^2 \leq 2\epsilon_0^{1/4}\|x\| \quad , \quad \sqrt{\epsilon/\lambda} \leq \epsilon_0^{1/4}\|x\|,$$

hence

$$\dot{x} \cdot x \geq 2\alpha\|x\|^2 - \|x\|^2 \epsilon_0^{1/4} O(\sqrt{\epsilon\lambda}) \geq \alpha\|x\|^2$$

provided  $\epsilon_0$  is small enough. Similarly,  $\dot{y} \cdot y \leq -\alpha\|y\|^2$  on  $B^u \times \partial B^s$  provided  $\epsilon_0$  is small enough. Concerning the linearized system, we have

$$\begin{aligned} L_{uu} &= \sqrt{\epsilon}\Lambda + O(\sqrt{\epsilon}\delta\lambda^{-1/4}\gamma^{-1} + \sqrt{\epsilon}\lambda^{-3/4}\rho + \epsilon\lambda^{-5/4} + \sqrt{\epsilon}\gamma) \\ &= \sqrt{\epsilon}\Lambda + O(\epsilon_0^{1/4}\sqrt{\epsilon\lambda}) \geq \alpha I, \\ L_{ss} &= -\sqrt{\epsilon}\Lambda + O(\epsilon_0^{1/4}\sqrt{\epsilon\lambda}) \leq -\alpha I \end{aligned}$$

on  $B^u \times B^s \times \Omega_\sigma$ . These inequalities holds when  $\epsilon_0$  is small enough because  $\sqrt{\epsilon}\Lambda \geq 2\alpha I$  and  $\sqrt{\epsilon\lambda} \leq O(\alpha)$ . Finally, still with the notations of Proposition B.3 [13], as in the previous estimate for  $L_{uu}$  we take

$$\begin{aligned} m &= O(\sqrt{\epsilon}\delta\lambda^{-1/4}\gamma^{-1} + \sqrt{\epsilon}\lambda^{-3/4}\rho + \epsilon\lambda^{-5/4} + \sqrt{\epsilon}\gamma + \sqrt{\epsilon}\delta/\sigma) \\ &= \sqrt{\epsilon\lambda} O(\sqrt{\delta}\lambda^{-3/4} + \rho\lambda^{-5/4} + \sqrt{\epsilon}\lambda^{-7/4}) = \sqrt{\epsilon\lambda} O(\epsilon_0^{1/4}). \end{aligned}$$

If  $\epsilon_0$  is small enough, we have  $4m < \alpha$  hence  $K \leq 2m/\alpha \leq O(\epsilon_0^{1/4}) < 2^{-1/2}$ , and Proposition B.3 [13] applies. The invariant strip obtained from the proof of this Proposition does not depend on the choice of  $\rho$ . It contains all the full orbits contained in

$$\{x : \|x\| \leq \epsilon_0^{1/4}\lambda^{-5/4}\} \times \{y : \|y\| \leq \epsilon_0^{1/4}\lambda^{-5/4}\} \times \mathbb{R} \times [M, 5\epsilon^{-1/3}] \times \mathbb{R}, \quad (24)$$

hence all the full orbits contained in  $V$ , as defined in the statement of Theorem 15. The possibility of taking  $\rho = \epsilon_0^{-1/4}(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\epsilon})$  now implies that the cylinder is actually contained in the domain where

$$\|x\|, \|y\| \leq \epsilon_0^{-1/4}(\lambda^{-3/4}\delta + \lambda^{-1/2}\sqrt{\epsilon}).$$

Moreover, with this choice of  $\rho$  and using the estimate for  $m$  we have that  $K = O(m/\sqrt{\epsilon\lambda}) = O(\epsilon_0^{1/4})$ .

Observe finally that, since the system is  $1/\gamma$ -periodic in  $\Theta$  and 1-periodic in  $t$ , so is the invariant strip that we obtain, as follows from Proposition B.2 [13]. We have obtained the existence of a  $C^1$  map

$$w^c = (w_u^c, w_s^c) : (\Theta, I, t) \in \mathbb{R} \times [M, 5\epsilon^{-1/3}] \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$$

which is  $2K$ -Lipschitz,  $1/\gamma$ -periodic in  $\Theta$  and 1-periodic in  $t$ , and the graph of which is weakly invariant.

Our last task is to return to the original coordinates by setting

$$\begin{aligned} \Theta^s(\theta^f, p^f, t) &= \theta_*^s(p^f) + \frac{1}{2}T(p^f) \cdot (w_u^c + w_s^c)(\gamma\theta^f, \epsilon^{-1/2}p^f, t) \\ P^s(\theta^f, p^f, t) &= p_*^s(p^f) + \frac{\sqrt{\epsilon}}{2}T^{-1}(p^f) \cdot (w_u^c - w_s^c)(\gamma\theta^f, \epsilon^{-1/2}p^f, t). \end{aligned}$$

All the estimates stated in Theorem 15 follow directly from these expressions, and from the fact that  $\|dw^c\| \leq 2K$ . This concludes the proof of Theorem 15.  $\square$



## 8 Proof of Key Theorem 3 about existence of invariant cylinders at double resonances

Key Theorem 3 follows from Theorem 7. Theorem 7, in turn, follows from Theorem 6.

The proof of Theorem 6 consists of two main parts:

- study properties of the local maps to establish hyperbolicity
- using hyperbolicity of the local map, construct a isolating block of Conley-McGehee [63] for various compositions of global and local maps.

Analysis of properties of the local map has three steps:

In section 8.1 we derive a finitely-smooth normal form in a neighborhood of the origin.

In section 8.2 we derive certain hyperbolic properties of the local map  $\Phi_{loc}^*$ , i.e. the map from a subset inside of the (incoming) section  $\Sigma_+^s$  to the (outgoing) section  $\Sigma_+^u$  (see Figure 9).

In section 8.3, using that eigenvalues are distinct, we establish strong hyperbolicity of the local map  $\Phi_{loc}^*$  as well as existence of unstable cones<sup>17</sup>. Since the global maps  $\Phi_{glob}^*$  have bounded time, they have bounded norms and the linearization of the proper compositions  $\Phi_{glob}^* \Phi_{loc}^*$  are dominated by the local component.

In section 8.4 we give definition and derive simple properties of isolating blocks of Conley-McGehee [63].

In section 8.5, under non-degeneracy conditions [A1]-[A4], we construct isolating blocks for the proper compositions of  $\Phi_{glob}^* \Phi_{loc}^*$ . This proves Theorem 6 for simple loops.

In section 8.6 we extend this analysis to  $\Phi_{glob}^* \Phi_{loc}^* \cdots \Phi_{glob}^* \Phi_{loc}^*$ . This would imply existence of families of shadowing orbits in non-simple case. This would prove Theorem 6.

In section 8.7 we complete a proof of Theorem 7 by showing that periodic orbits constructed in Theorem 6 are hyperbolic and their union forms a normally hyperbolic invariant cylinder.

Now we assume that the Hamiltonian  $H_\varepsilon$  is  $C^{k+1}$  with  $k \geq 9$ . In Section 8.8 using approximation arguments we remove this condition.

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<sup>17</sup>One can expect this as near the origin eigenvalues  $(-\lambda_2 < -\lambda_1 < 0 < \lambda_1 < \lambda_2)$  dominate the linearization of the flow and provide hyperbolicity. The closer orbits pass to the origin the stronger hyperbolicity of the local map  $\Phi_{loc}^*$ .

## 8.1 Normal form near the hyperbolic fixed point

We describe a normal form near the hyperbolic fixed point (assumed to be  $(0, 0)$ ) of the slow Hamiltonian  $H^s : \mathbb{T}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ . For the rest of this section, we drop the subscript  $s$  to abbreviate notations. In a neighborhood of the origin, there exists a symplectic linear change of coordinates under which the system has the normal form

$$H(u_1, u_2, s_1, s_2) = \lambda_1 s_1 u_1 + \lambda_2 s_2 u_2 + O_3(s, u).$$

Here  $s = (s_1, s_2)$ ,  $u = (u_1, u_2)$ , and  $O_n(s, u)$  stands for a function bounded by  $C|(s, u)|^n$ .

The main result of this section is the following improved normal form

**Theorem 18.** *Let  $H$  be  $C^{k+1}$  with  $k \geq 9$ , then there exists neighborhood  $U$  of the origin,  $m = m(\lambda_2, \lambda_1, k)$ , and a  $C^2$  change of coordinates  $\Phi$  on  $U$  such that  $N_m = H \circ \Phi$  is a polynomial of degree  $m$  of the form*

$$\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -\partial_{u_1} N_m \\ -\partial_{u_2} N_m \\ \partial_{s_1} N_m \\ \partial_{s_2} N_m \end{bmatrix} = \begin{bmatrix} -\lambda_1 s_1 + F_1(s, u) \\ -\lambda_2 s_2 + F_2(s, u) \\ \lambda u_1 + G_1(s, u) \\ \lambda u_2 + G_2(s, u) \end{bmatrix} \quad (25)$$

where

$$\begin{aligned} F_1 &= s_1 O_1(s, u) + s_2 O_1(s, u), & F_2 &= s_1^2 O(1) + s_2 O_1(s, u), \\ G_1 &= u_1 O_1(s, u) + u_2 O_1(s, u), & G_2 &= u_1^2 O(1) + u_2 O_1(s, u). \end{aligned}$$

The proof consists of two steps: first, we do some preliminary normal form and then apply a theorem of Belitskii-Samovol (see, for example, [22]).

Since  $(0, 0)$  is a hyperbolic fixed point, for sufficiently small  $r > 0$ , there exists stable manifold  $W^s = \{(u = U(s), |s| \leq r)\}$  and unstable manifold  $W^u = \{s = S(u), |u| \leq r\}$  containing the origin. All points on  $W^s$  converges to  $(0, 0)$  exponentially in forward time, while all points on  $W^u$  converges to  $(0, 0)$  exponentially in backward time. These manifolds are Lagrangian; as a consequence, the change of coordinates  $s' = s - S(u)$ ,  $u' = u - U(s') = u - U(s - S(u))$  is symplectic. Under the new coordinates, we have that  $W^s = \{u' = 0\}$  and  $W^u = \{s' = 0\}$ . We abuse notation and keep using  $(s, u)$  to denote the new coordinate system.

Under the new coordinate system, the Hamiltonian has the form

$$H(s, u) = \lambda_1 s_1 u_1 + \lambda_2 s_2 u_2 + H_1(s, u),$$

where  $H(s, u) = O_3(s, u)$  and  $H_1(s, u)|_{s=0} = H_1(s, u)|_{u=0} = 0$ . Let us denote  $H_0 = \lambda_1 s_1 u_1 + \lambda_2 s_2 u_2$ . We now perform a further step of normalization.

We say an tuple  $(\alpha, \beta) \in \mathbb{N}^2 \times \mathbb{N}^2$  is *resonant* if  $\sum_{i=1}^2 \lambda_i(\alpha_i - \beta_i) = 0$ . Note that an  $(\alpha, \beta)$  with  $\alpha_i = \beta_i$  for  $i = 1, 2$  is always resonant. A monomial  $u_1^{\alpha_1} u_2^{\alpha_2} s_1^{\beta_1} s_2^{\beta_2}$  is resonant if  $(\alpha, \beta)$  is resonant. Otherwise, we call it *nonresonant*. It is well known that a Hamiltonian can always be transformed, via a formal power series, to an Hamiltonian with only resonant terms (see e.g. [72], section 30, for example). We do not use Hamiltonian structure of the flow. Thus, it suffices to have an analogous claim for vector fields with a coordinate change being only finitely smooth. For a complex  $\mu \in \mathbb{C}$  denote  $\Re \mu$  the real part of  $\mu$ .

**Theorem 19** (Belitskii-Samovol). [22] *Let  $k$  be positive integer. Assume that the vector field  $\dot{x} = F(x)$  is of class  $C^K$ ,  $x = 0$  is a hyperbolic saddle point  $F(0) = 0$  and  $A = DF(0)$  is the linearization. Let  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  be the spectrum of  $A$ . Suppose real parts of  $\lambda_i$ 's are all nonzero and pairwise disjoint. If  $K \geq dk + 1$ , then for some positive integer  $m$ , this vector field near the point 0 by a transformation  $y = \Phi(x)$ ,  $\Phi \in C^k$ , can be reduced to the polynomial resonant normal form*

$$\dot{y} = Ay + \sum_{|\tau|=2}^m p_\tau y^\tau,$$

where  $\tau \in \mathbb{Z}_+^n$  and  $p_\tau$  denotes vector coefficients of a multi-homogeneous polynomial  $p_\tau = (p_\tau^1, \dots, p_\tau^n)$  and  $p_\tau^i \neq 0$  for some  $i = 1, \dots, n$  implies  $\Re \lambda_i = \tau^1 \Re \lambda_1 + \dots + \tau^n \Re \lambda_n$  (by the resonant condition).

In [22] there is an upper bound on  $m$ . One can also find  $\lambda$ -dependent lower bounds on smoothness exponent  $K$  there.

Application of this Theorem with  $n = 4$ ,  $k = 2$ ,  $K = 9$ ,  $\lambda = (-\lambda_2, -\lambda_1, \lambda_1, \lambda_2)$ ,  $0 < \lambda_1 < \lambda_2$  gives existence of a  $C^2$ -change of coordinates  $\Phi$  such that  $N_m = H \circ \Phi$  consists of only resonant monomials.

We abuse notations by replacing  $(s', u')$  with  $(s, u)$ . Using our assumption that  $0 < \lambda_1 < \lambda_2$ , we have that all  $(\alpha, \beta)$  with  $\alpha \neq \beta$ ,  $\alpha_1 = 1$  and  $\alpha_2 = 0$  are nonresonant, and similarly, all  $(\alpha, \beta)$  with  $\alpha \neq \beta$ ,  $\beta_1 = 1$  and  $\beta_2 = 0$  are nonresonant. Furthermore, by performing the straightening of stable/unstable manifolds again if necessary, we may assume that  $N_m|_{s=0} = N_m|_{u=0} = 0$ . As a consequence, the normal form  $N_m$  must take the following form:

**Corollary 8.1.** *The normal form  $N_m$  satisfies*

$$\begin{aligned} N_m = & \lambda_1 s_1 u_1 + \lambda_2 s_2 u_2 + \\ & + O_1(u_1^2 s_2) + O_1(s_1^2 u_1^2) + O_1(s_1^2 u_2) + O_1(s_2 u_1^2) + O_1(s_1 u_1 s_2 u_2) + O_1(s_2^2 u_2^2). \end{aligned}$$

*In particular, we have  $N_m = \lambda_1 s_1 u_1 + \lambda_2 s_2 u_2 + O_3(s, u)$ .*

Explicit differentiation of the remainder terms implies the form of partial derivatives of  $N_m$  given by  $F_1, F_2, G_1, G_2$  in Theorem 18. Notice that bounds are not optimal, but sufficient for our purposes.

## 8.2 Behavior of a family of orbits passing near 0 and Shil'nikov boundary value problem

The main result of this section is the following

**Theorem 20.** *Let  $(s^T, u^T)$  be a family of orbits satisfying  $s^T(0) \rightarrow s^{in}$  as  $T \rightarrow \infty$  with  $s_1^T = \delta$  and  $u^T(T) \rightarrow u^{out}$  as  $T \rightarrow \infty$  with  $s_1^T = \delta$  with  $|s^T|, |u^T| \leq 2\delta$ , where  $\delta$  is small enough. Then there exists  $T_0, C > 0$  and  $\alpha > 1$  such that for each  $T > T_0$  and all  $0 \leq t \leq T$  we have*

$$|s_2^T(t)| \leq C|s_1^T(t)|^\alpha, \quad |u_2^T(t)| \leq C|u_1^T(t)|^\alpha.$$

*In particular, the curve  $\{(s_1^T(T), s_2^T(T))\}_{T \geq T_0} \subset \Sigma_+^u = \{s_1^T(0) = \delta\}$  is tangent to the  $s_1$ -axis at  $T = \infty$  and  $\{(u_1^T(0), u_2^T(0))\}_{T \geq T_0} \subset \Sigma_+^s = \{s_1^T(0) = \delta\}$  is tangent to the  $u_1$ -axis at  $T = \infty$ .*

We will use the local normal form to study the local maps. Our main technical tool to prove the above Theorem is the following *boundary value problem due to Shil'nikov* (see [70]):

**Proposition 8.2.** *There exists  $\kappa_0 > 0$  such that for any  $0 < \kappa \leq \kappa_0$ , there exist  $\delta > 0$  such that the following hold. For any  $s^{in} = (s_1^{in}, s_2^{in})$ ,  $u^{out} = (u_1^{out}, u_2^{out})$  with  $|s|, |u| \leq \delta$  and any large  $T > 0$ , there exists a unique solution  $(s^T, u^T) : [0, T] \rightarrow B_\delta$  of the system (25) with the property  $s^T(0) = s^{in}$  and  $u^T(T) = u^{out}$ . Let*

$$(s^{(1)}, u^{(1)})(t) = (e^{-\lambda_1 t} s_1^{in}, e^{-\lambda_2 t} s_2^{in}, e^{-\lambda_1(T-t)} u_1^{out}, e^{-\lambda_2(T-t)} u_2^{out}), \quad (26)$$

*we have*

$$\begin{aligned} |s_1^T(t) - s_1^{(1)}(t)| &\leq \delta e^{-(\lambda_1 - \kappa)t}, & |s_2^T(t) - s_2^{(1)}(t)| &\leq \delta e^{-(\lambda'_2 - 2\kappa)t}, \\ |u_1^T(t) - u_1^{(1)}(t)| &\leq \delta e^{-(\lambda_1 - \kappa)(T-t)}, & |u_2^T(t) - u_2^{(1)}(t)| &\leq \delta e^{-(\lambda'_2 - 2\kappa)(T-t)}, \end{aligned}$$

*where  $\lambda'_2 = \min\{\lambda_2, 2\lambda_1\}$ . Furthermore, for  $s_1$  and  $u_1$ , we have an additional lower bound estimate:*

$$|s_1^T(t)| \geq \frac{1}{2} |s_1^{in}| e^{-(\lambda_1 + \kappa)t}, \quad |u_1^T(t)| \geq \frac{1}{2} |u_1^{out}| e^{-(\lambda_1 + \kappa)(T-t)}. \quad (27)$$

*Note that for (27) to hold, the choice of  $\delta$  needs to depend on a lower bound for  $|s_1^{in}|$  and  $|u_1^{out}|$ .*

*Proof.* Let  $\Gamma$  denote the set of all smooth curves  $(s, u) : [0, T] \longrightarrow B(0, \delta)$  such that the  $s(0) = (s_1^{in}, s_2^{in})$  and  $u(T) = (u_1^{out}, u_2^{out})$ . We define a map  $\mathcal{F} : \Gamma \longrightarrow \Gamma$  by  $\mathcal{F}(s, u) = (\tilde{s}, \tilde{u})$ , where

$$\begin{aligned}\tilde{s}_1 &= e^{-\lambda_1 t} s_1^{in} + \int_0^t e^{\lambda_1(\xi-t)} F_1(s(\xi), u(\xi)) d\xi, \\ \tilde{s}_2 &= e^{-\lambda_2 t} s_2^{in} + \int_0^t e^{\lambda_2(\xi-t)} F_2(s(\xi), u(\xi)) d\xi, \\ \tilde{u}_1 &= e^{-\lambda_1(T-t)} u_1^{out} - \int_t^T e^{-\lambda_1(\xi-t)} G_1(s(\xi), u(\xi)) d\xi, \\ \tilde{u}_2 &= e^{-\lambda_2(T-t)} u_2^{out} - \int_t^T e^{-\lambda_2(\xi-t)} G_2(s(\xi), u(\xi)) d\xi.\end{aligned}$$

It is proved in [70] that for sufficiently small  $\delta$ , the map  $\mathcal{F}$  is a contraction in the uniform norm. Let  $s^{(1)}, u^{(1)}$  be as defined in (26) and  $(s^{(k+1)}, u^{(k+1)}) = \mathcal{F}(s^{(k)}, u^{(k)})$ , then  $(s^{(k)}, u^{(k)})$  converges to the solution of the boundary value problem. Using the normal form (25), we will provide precise estimates on the sequence  $(s^{(k)}, u^{(k)})$ . The upper bound estimates are consequences of the following:

$$\begin{aligned}|s_1^{(k+1)}(t) - s_1^{(k)}(t)| &\leq 2^{-k} \delta e^{-(\lambda_1 - \kappa)t}, \\ |s_2^{(k+1)}(t) - s_2^{(k)}(t)| &\leq 2^{-k} \delta e^{-(\lambda'_2 - \kappa)t}, \\ |u_1^{(k+1)}(t) - u_1^{(k)}(t)| &\leq 2^{-k} \delta e^{-(\lambda_1 - \kappa)(T-t)}, \\ |u_2^{(k+1)}(t) - u_2^{(k)}(t)| &\leq 2^{-k} \delta e^{-(\lambda'_2 - \kappa)(T-t)}.\end{aligned}$$

We have

$$\begin{aligned}|s_1^{(2)}(t) - s_1^{(1)}(t)| &= \int_0^t e^{\lambda_1(\xi-t)} \left| s_1^{(1)}(\xi) O_1(s, u) + s_2^{(1)}(\xi) O_1(s, u) \right| d\xi \\ &\leq \int_0^T e^{\lambda_1(\xi-t)} (O(\delta^2) e^{-\lambda_1 \xi} + O(\delta^2) e^{-\lambda_2 \xi}) d\xi \\ &\leq O(\delta^2) t e^{-\lambda_1 t} \leq C \frac{t e^{\varepsilon t}}{\kappa t} \delta^2 e^{-(\lambda_1 - \kappa)t} \leq C \kappa^{-1} \delta^2 e^{-(\lambda_1 - \kappa)t} \leq \frac{1}{2} \delta e^{-(\lambda_1 - \kappa)t}.\end{aligned}$$

Note that the last inequality can be guaranteed by choosing  $\delta \leq C^{-1}\kappa$ . Similarly

$$\begin{aligned}
|s_2^{(2)}(t) - s_2^{(1)}(t)| &= \int_0^t e^{\lambda_2(\xi-t)} \left| (s_1^{(1)}(\xi))^2 O(1) + s_2^{(1)}(\xi) O_1(s, u) \right| d\xi \\
&\leq \int_0^t e^{\lambda_2(\xi-t)} (O(\delta^2) e^{-2\lambda_1\xi} + O(\delta^2) e^{-\lambda_2\xi}) d\xi \\
&\leq O(\delta^2) \int_0^t e^{\lambda_2'(\xi-t)} e^{-\lambda_2'\xi} d\xi \leq C\delta^2 t e^{-\lambda_2't} \leq C\delta^2 \frac{e^{2\varepsilon t}}{2\kappa} e^{-\lambda_2't} \\
&\leq C\kappa^{-1} \delta^2 e^{-(\lambda_2'-\kappa)t} \leq \frac{1}{2} \delta e^{-(\lambda_2'-2\kappa)t}.
\end{aligned}$$

Observe that the calculations for  $u_1$  and  $u_2$  are identical if we replace  $t$  with  $T - t$ . We obtain

$$|u_1^{(2)}(t) - u_1^{(1)}(t)| \leq \frac{1}{2} \delta e^{-(\lambda_1-\kappa)(T-t)}, \quad |u_2^{(2)}(t) - u_2^{(1)}(t)| \leq \frac{1}{2} \delta e^{-(\lambda_2'-2\kappa)(T-t)}.$$

According to the normal form (25), we have there exists  $C' > 0$  such that

$$\|\partial_s F_1\| \leq C' \|(s, u)\|, \quad \|\partial_u F_1\| \leq C' \|s\|.$$

Using the inductive hypothesis for step  $k$ , we have  $\|s^{(k)}(t)\| \leq 2\delta e^{-(\lambda_1-\kappa)t}$ . It follows that

$$\begin{aligned}
|s_1^{(k+2)}(t) - s_1^{(k+1)}(t)| &\leq \int_0^t e^{\lambda_1(\xi-t)} (\|\partial_s F_1\| \|s^{(k+1)} - s^{(k)}\| + \|\partial_u F_1\| \|u^{(k+1)} - u^{(k)}\|) d\xi \\
&\leq C' \int_0^t e^{\lambda_1(\xi-t)} (\delta 2^{-k} \delta e^{-(\lambda_1-\kappa)\xi} + \delta e^{-(\lambda_1-\kappa)\xi} 2^{-k} \delta) d\xi \\
&\leq 2^{-k} \delta e^{-(\lambda_1-\kappa)t} \int_0^t 2C' e^{-\kappa\xi} \delta d\xi \leq 2^{-(k+1)} \delta e^{-(\lambda_1-\kappa)t}.
\end{aligned}$$

Note that the last inequality can be guaranteed by choosing  $\delta$  sufficiently small depending on  $C'$  and  $\kappa$ . The estimates for  $s_2$  needs more detailed analysis. We write

$$\begin{aligned}
|s_2^{(k+2)}(t) - s_2^{(k+1)}(t)| &\leq \int_0^t e^{\lambda_2(\xi-t)} \cdot \\
&\quad \left( \|\partial_{s_1} F_2\| |s_1^{(k+1)} - s_1^{(k)}| + \|\partial_{s_2} F_2\| |s_2^{(k+1)} - s_2^{(k)}| + \|\partial_u F_2\| \|u_2^{(k+1)} - u_2^{(k)}\| \right) d\xi \\
&= \int_0^t e^{\lambda_2(\xi-t)} (I + II + III) d\xi.
\end{aligned}$$

We have  $\|\partial_{s_1} F_2\| = O_1(s_1)O(1) + O_1(s_2)O(1)$ , hence

$$I \leq C'(\delta e^{-(\lambda_1 - \kappa)\xi} + \delta e^{-(\lambda'_2 - 2\kappa)\xi})2^{-k}\delta e^{-(\lambda_1 - \kappa)\xi} \leq C'2^{-k}\delta 2e^{-2(\lambda'_1 - \kappa)\xi}.$$

Since  $\|\partial_{s_2} F_2\| = O_2(s_1) + O_1(s, u) = O_1(s, u)$ , we have  $II \leq C'\delta^2 2^{-k}e^{-(\lambda'_2 - 2\kappa)\xi}$ . Finally, as  $\|\partial_u F_2\| = O_2(s_1) + O_1(s_2)O(1)$ , we have

$$III \leq C'2^{-k}\delta(\delta^2 e^{-2(\lambda_1 - \kappa)\xi} + \delta e^{-(\lambda'_2 - 2\kappa)\xi}) \leq C'2^{-k}\delta^2 e^{-(\lambda'_2 - 2\kappa)\xi}.$$

Note that in the last line, we used  $\lambda'_2 \leq 2\lambda_1$ . Combine the estimates obtained, we have

$$\begin{aligned} |s_2^{(k+2)}(t) - s_2^{(k+1)}(t)| &\leq \delta 2^{-k} \int_0^t 3C'\delta e^{\lambda_2(\xi-t)} e^{-(\lambda'_2 - 2\kappa)\xi} d\xi \\ &\leq \delta 2^{-k} e^{-(\lambda'_2 - 2\kappa)t} \int_0^t 3C'\delta e^{-2\kappa\xi} d\xi \leq 2^{-(k+1)} \delta e^{-(\lambda'_2 - 2\kappa)t}. \end{aligned}$$

The estimates for  $u_1$  and  $u_2$  follow from symmetry.

We now prove the lower bound estimates (27). We will first prove the estimates for  $s_1$  in the case of  $s_1^{in} > 0$ . We have the following differential inequality

$$\dot{s}_1 \geq -(\lambda_1 + C'\delta)s_1 + s_2 O_1(s, u).$$

Note that  $|s_2(t)| \leq 2\delta e^{-\lambda'_2 t}$  due to the already established upper bound estimates. Choose  $\delta$  such that  $C'\delta \leq \kappa$ , we have

$$\begin{aligned} s_1(t) &\geq s_1^{in} e^{-(\lambda_1 + \kappa)t} - \int_0^t e^{-(\lambda_1 + \kappa)(\xi-t)} 2\delta e^{-(\lambda'_2 - 2\kappa)\xi} \cdot C'\delta d\xi \\ &\geq s_1^{in} e^{-(\lambda_1 + \kappa)t} - 2C'\delta^2 (\lambda'_2 - \lambda_1 - 3\kappa)^{-1} e^{-(\lambda_1 + \kappa)t} \geq \frac{1}{2} s_1^{in} e^{-(\lambda_1 + \kappa)t}. \end{aligned}$$

For the last inequality to hold, we choose  $\kappa_0$  small enough such that  $\lambda'_2 - \lambda_1 - 3\kappa > 0$ , and choose  $\delta$  such that  $2C'\delta^2 (\lambda'_2 - \lambda_1 - 3\kappa)^{-1} \leq \frac{1}{2} s_1^{in}$ .

The case when  $s_1^{in} < 0$  follows from applying the above analysis to  $-s_1$ . The estimates for  $u_1$  can be obtained by replacing  $s_i$  with  $u_i$  and  $t$  with  $T - t$  in the above analysis.  $\square$

*Proof of Theorem 20.* The following estimate follows from Proposition 8.2 that  $|s_1^T(t)| \geq \frac{1}{2} |s_1^{in}| e^{-(\lambda_1 + \kappa)t}$  and  $|s_2^T(t)| \leq 2\delta e^{-(\lambda'_2 - 2\kappa)t}$ . We obtain the estimates for  $s_1$  and  $s_2$  by choosing  $\alpha = \frac{\lambda_2 - 2\epsilon}{\lambda_1 + \kappa}$  and  $C = 4\delta/|s_1^{in}|$ . The case of  $u_1$  and  $u_2$  can be proved similarly.  $\square$

### 8.3 Properties of the local maps

Recall that  $\gamma^+$  is a homoclinic orbit satisfying the conditions [A1]-[A4], and  $\gamma^-$  is its time-reversal. Denote  $p^\pm = (s^\pm, 0) = \gamma^\pm \cap \Sigma_\pm^s$  and  $q^\pm = (0, u^\pm) = \gamma^\pm \cap \Sigma_\pm^u$ . Although the local map  $\Phi_{\text{loc}}^{++}$  is not defined at  $p^+$  (and its inverse is not defined at  $q^+$ ), the map is well defined from a neighborhood close to  $p^+$  to a neighborhood close to  $q^+$ . In particular, for any  $T > 0$ , by Proposition 8.2, there exists a trajectory  $(s, u)_T^{++}$  of the Hamiltonian flow such that

$$s_T^{++}(0) = s^+, \quad u_T^{++}(T) = u^+.$$

Denote  $x_T^{++} = (s, u)_T^{++}(0)$  and  $y_T^{++} = (s, u)_T^{++}(T)$ , we have  $\Phi_{\text{loc}}^{++}(x_T^{++}) = y_T^{++}$ , and  $x_T^{++} \rightarrow p^+$ ,  $y_T^{++} \rightarrow q^+$  as  $T \rightarrow \infty$ . We apply the same procedure to other local maps and extend the notations by changing the superscripts accordingly.

Let  $N = N_k(s, u)$  be the Hamiltonian in the normal form from Theorem 18,  $E(T) = N((s, u)_T^{++})$  be the energy of the orbit, and  $S_{E(T)} = \{N = E(T)\}$  be the corresponding energy surface. We will show that the domain of  $\Phi_{\text{loc}}^{++}|_{S_{E(T)}}$  can be extended to a larger subset of  $\Sigma_+^{s, E(T)}$  containing  $x_T^{++}$ . We call  $R \subset \Sigma_+^s \cap S_{E(T)}$  a rectangle if it is bounded by four vertices  $x_1, \dots, x_4$  and  $C^1$  curves  $\gamma_{ij}$  connecting  $x_i$  and  $x_j$ , where  $ij \in \{12, 34, 13, 24\}$ . The curves does not intersect except at the vertices. Denote  $B_\delta(x)$  the  $\delta$ -ball around  $x$  and the local parts of invariant manifolds

$$\Sigma_s^+ = W^s(0) \cap \Sigma_+^s \cap B_\delta(p^+), \quad \Sigma_u^+ = W^u(0) \cap \Sigma_+^u \cap B_\delta(q^+)$$

and the  $\Sigma$ -sections restricted to an energy surface  $S_E$  by

$$\Sigma_+^{s, E} = \Sigma_+^s \cap S_E \quad \text{and} \quad \Sigma_+^{u, E} = \Sigma_+^u \cap S_E.$$

The main result of this section is the following

**Theorem 21.** *There exists  $\delta_0 > 0$  and  $T_0 > 0$  such that for any  $T > T_0$  and  $0 < \delta < \delta_0$ , there exists a rectangle  $R^{++}(T) \subset \Sigma_+^{s, E(T)}$ , with vertices  $x_i(T)$  and  $C^1$ -smooth sides  $\gamma_{ij}(T)$ , such that the following hold:*

1.  $\Phi_{\text{loc}}^{++}$  is well defined on  $R^{++}(T)$ .  $\Phi_{\text{loc}}^{++}(R^{++}(T))$  is also a rectangle with vertices  $x'_i(T)$  and sides  $\gamma'_{ij}(T)$ .
2. As  $T \rightarrow 0$ ,  $\gamma_{12}(T)$  and  $\gamma_{34}(T)$  both converge in Hausdorff metric to a single curve containing  $\Sigma_s^+$ ;  $\gamma'_{13}(T)$  and  $\gamma'_{24}(T)$  converges to a single curve containing  $\Sigma_u^+$ .

*The same conclusions, after substituting the superscripts according to the signatures of the map, hold for other local maps.*



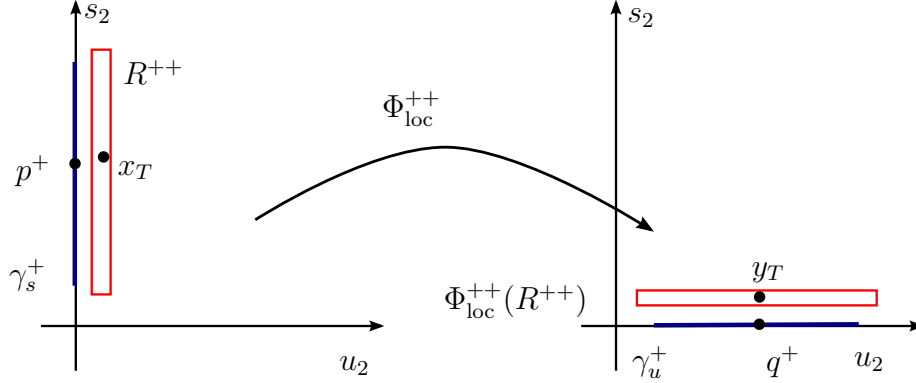


Figure 18: Local map  $\Phi_{loc}^{++}$

To get a picture of Theorem 21, note that for a given energy  $E > 0$ , the restricted sections  $\Sigma_+^{s,E}$  and  $\Sigma_+^{u,E}$  are both transversal to the  $s_1$  and  $u_1$  axes, and hence these sections can be parametrized by the  $s_2$  and  $u_2$  components. An illustration of the local maps and the rectangles is contained in Figure 18 and 19.

We will only prove Theorem 21 for the local map  $\Phi_{loc}^{++}$ . The proof for the other local maps are identical with proper changes of notations.

Let  $(v_{s_1}, v_{s_2}, v_{u_1}, v_{u_2})$  denote the coordinates for the tangent space induced by  $(s_1, s_2, u_1, u_2)$ . As before  $B_r$  denotes the  $r$ -neighborhood of the origin. For  $c > 0$  and  $x \in B_r$ , we define the *strong unstable cone* by

$$C^{u,c}(x) = \{c|v_{u_2}|^2 > |v_{u_1}|^2 + |v_{s_1}|^2 + |v_{s_2}|^2\}$$

and the *strong stable cone* to be

$$C^{s,c}(x) = \{c|v_{s_2}|^2 > |v_{s_1}|^2 + |v_{u_1}|^2 + |v_{u_2}|^2\}.$$

The following properties follows from the fact that the linearization of the flow at 0 is hyperbolic. We will drop the superscript  $c$  when the dependence in  $c$  is not stressed.

**Lemma 8.3.** *For any  $0 < \kappa < \lambda_2 - \lambda_1$ , there exists  $r = r(\kappa, c)$  such that the following holds:*

- *If  $\varphi_t(x) \in B_r$  for  $0 \leq t \leq t_0$ , then  $D\varphi_t(C^u(x)) \subset C^u(\varphi_t(x))$  for all  $0 \leq t \leq t_0$ . Furthermore, for any  $v \in C^u(x)$ ,*

$$|D\varphi_t(x)v| \geq e^{(\lambda_2 - \kappa)t}, \quad 0 \leq t \leq t_0.$$

- *If  $\varphi_{-t}(x) \in B_r$  for  $0 \leq t \leq t_0$ , then  $D\varphi_{-t}(C^s(x)) \subset C^s(\varphi_{-t}(x))$  for all  $0 \leq t \leq t_0$ . Furthermore, for any  $v \in C^s(x)$ ,*

$$|D\varphi_{-t}(x)v| \geq e^{(\lambda_2 - \kappa)t}, \quad 0 \leq t \leq t_0.$$

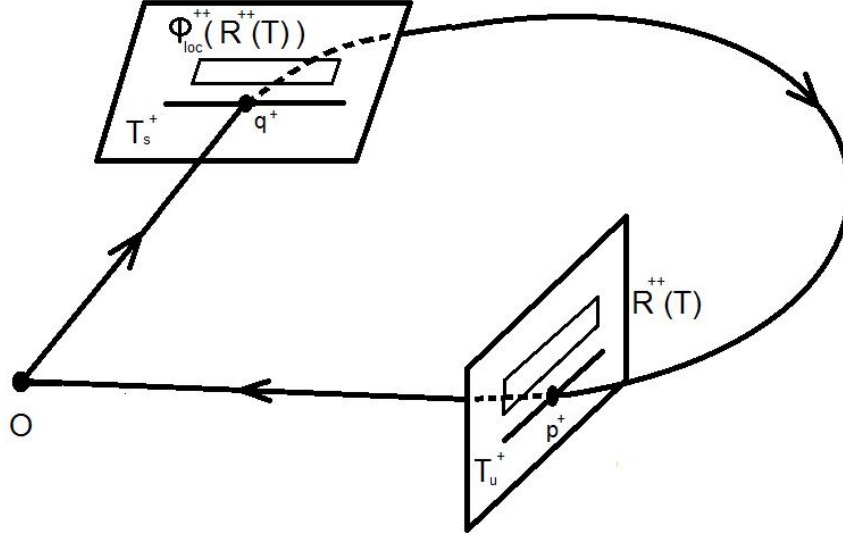


Figure 19: Rectangles mapped under  $\Phi_{loc}^{++}$

For each energy surface  $E$ , we define the restricted cones  $C_E^u(x) = C^u(x) \cap T_x S_E$  and  $C_E^s(x) = C^s(x) \cap T_x S_E$ .

**Warning:** Recall that the Hamiltonian  $N$  under consideration by Theorem 18 has the form  $N_k = \lambda_1 s_1 u_1 + \lambda_2 s_2 u_2 + O_3(s, u)$ . It is easy to see that the restricted cones  $C_E^u(x)$  and  $C_E^s(x)$  might be *empty*. Excluding this case requires special care!

Since the energy surface is invariant under the flow, its tangent space is also invariant. We have the following observation:

**Lemma 8.4.** *If  $\varphi_t(x) \in B_r$  for  $0 \leq t \leq t_0$ , then  $C_E^u$  is invariant under the map  $D\varphi_t$  for  $0 \leq t \leq t_0$ . In particular, if  $C_E^u(x) \neq \emptyset$ , then  $C_E^u(\varphi_t(x)) \neq \emptyset$ . Similar conclusions hold for  $C_E^s$  with  $\varphi_{-t}$ .*

Let  $x$  be such that  $\varphi_t(x) \in B_r \cap S_E$  for  $0 \leq t \leq t_0$ . A Lipschitz curve  $\gamma_E^s(x)$  is called *stable* if its forward image stays in  $B_r$  for  $0 \leq t \leq t_0$ , and that the curve and all its forward images are tangent to the restricted stable cone field  $\{C_E^s\}$ . For  $y$  such that  $\varphi_{-t}(y) \in B_r \cap S_E$  for  $0 \leq t \leq t_0$ , we may define the *unstable curve*  $\gamma_E^u(y)$  in the same way with  $t$  replaced by  $-t$  and  $C_E^s$  replaced by  $C_E^u$ . Notice that stable and unstable curves are *not* in the tangent space, but in the phase space.

**Proposition 8.5.** *In notations of Lemma 8.3 assume that  $x, y \in S_E$  satisfies the following conditions.*

- $\varphi_t(x) \in B_r \cap S_E$  and  $\varphi_{-t}(y) \in B_r \cap S_E$  for  $0 \leq t \leq t_0$ .
- The restricted cone fields are not empty. Moreover, there exists  $a > 0$  such that  $C_E^{s,c}(\varphi_{t_0}(z)) \neq \emptyset$  for  $z \in U_a(\varphi_{t_0}(x)) \cap S_E$ , and  $C_E^{u,c}(\varphi_{-t_0}(z')) \neq \emptyset$  for each  $z' \in U_a(\varphi_{-t_0}(y)) \cap S_E$ .

Then there exists at least one stable curve  $\gamma_E^s(x)$  and one unstable curve  $\gamma_E^u(y)$ .

If  $a \geq \sqrt{c^2 + 1} re^{-(\lambda_2 - \kappa)t_0}$ , then the stable curve  $\gamma_E^s(x)$  and the unstable one  $\gamma_E^u(y)$  can be extended to the boundary of  $B_r(x)$  and of  $B_r(y)$  respectively. Furthermore,

$$\|\varphi_t(x) - \varphi_t(x_1)\| \leq e^{-(\lambda_2 - \kappa)t}, \quad x_1 \in \gamma_E^s(x), \quad 0 \leq t \leq t_0$$

and

$$\|\varphi_{-t}(y) - \varphi_{-t}(y_1)\| \leq e^{-(\lambda_2 - \kappa)t}, \quad y_1 \in \gamma_E^u(y), \quad 0 \leq t \leq t_0.$$

It is possible to choose the curves to be  $C^1$ .

**Remark 8.1.** The stable and unstable curves are not unique. Locally, there exists a cone family such that any curve tangent to this cone family is a stable/unstable curve.

*Proof.* Let us denote  $x' = \varphi_{t_0}(x)$ . From the smoothness of the flow, we have that there exist neighborhoods  $U$  of  $x$  and  $U'$  of  $x'$  such that  $\varphi_{t_0}(U) = U'$  and  $\varphi_t(U) \in B_r$  for all  $0 \leq t \leq t_0$ . By intersecting  $U'$  with  $U_a(x')$  if necessary, we may assume that  $U' \subset U_a(x')$ . We have that  $C_E^{s,c}(z) \neq \emptyset$  for all  $z \in U'$ . It then follows that there exists a curve  $\gamma_E^s(x') \subset U'$  that is tangent to  $C_E^{s,c}$ . As  $C_E^{s,c}$  is backward invariant with respect to the flow, we have that  $\varphi_{-t}(\gamma_E^s(x'))$  is also tangent to  $C_E^{s,c}$  for  $0 \leq t \leq t_0$ . Let  $\text{dist}(\gamma_E^s)$  denote the length of the curve  $\gamma_E^s$  and let  $\gamma_E^s(x) = \varphi_{-t_0}(\gamma_E^s(x'))$ . It follows from the properties of the cone field that

$$\text{dist}(\gamma_E^s(x)) \geq e^{(\lambda_2 - \kappa)t_0} \text{dist}(\gamma_E^s(x')).$$

We also remark that from the fact that  $\gamma_E^s(x)$  is tangent to the cone field  $C_E^{s,c}(x)$ , the Euclidean diameter (the largest Euclidean distance between two points) of  $\gamma_E^s(x)$  is bounded by  $\frac{1}{\sqrt{c^2 + 1}} \text{dist}(\gamma_E^s(x))$  from below and by  $l(\gamma_E^s(x))$  from above.

Let  $x_1$  be one of the end points of  $\gamma_E^s(x)$  and  $x'_1 = \varphi_{t_0}(x_1)$ . We may apply the same arguments to  $x_1$  and  $x'_1$ , and extend the curves  $\gamma_E^s(x)$  and  $\gamma_E^s(x')$  beyond  $x_1$  and  $x'_1$ , unless either  $x_1 \in \partial B_r$  or  $x'_1 \in \partial U_a(x')$ . This extension can be made keeping the  $C^1$  smoothness of  $\gamma$ . Denote  $\gamma_E^s(x)[x, x_1]$  the segment on  $\gamma_E^s(x)$  from  $x$  to  $x_1$ . We have that

$$\begin{aligned} \|x'_1 - x'\| &\leq \text{dist}(\gamma_E^s(x'))[x', x'_1] \leq \\ &\leq e^{-(\lambda_2 - \kappa)t_0} \text{dist}(\gamma_E^s(x))[x, x_1] \leq e^{-(\lambda_2 - \kappa)t_0} \|x - x_1\| \sqrt{c^2 + 1}. \end{aligned}$$

It follows that if  $a \geq r\sqrt{c^2 + 1}e^{-(\lambda_2 - \kappa)t_0}$ ,  $x_1$  will always reach boundary of  $B_r$  before  $x'_1$  reaches the boundary of  $U_a(x')$ . This proves that the stable curve can be extended to the boundary of  $B_r$ .

The estimate  $\|\varphi_t(x) - \varphi_t(x_1)\| \leq e^{-(\lambda_2 - \kappa)t}$  follows directly from the earlier estimate of the arc-length. This concludes our proof of the proposition for stable curves. The proof for unstable curves follows from the same argument, but with  $C_E^{s,c}$  replaced by  $C_E^{u,c}$  and  $t$  by  $-t$ .  $\square$

In order to apply Proposition 8.5 to the local map, we need to show that the restricted cone fields are not empty. (see also the warning after Lemma 8.3)

**Lemma 8.6.** *There exists  $0 < a \leq \delta$  and  $c > 0$  such that for any  $x = (s, u) \in \Sigma_+^{s,E}$  with  $\|u\| \leq a$ , and  $|s_2| \leq 2\delta$ , we have  $C_E^{u,c}(x) \neq \emptyset$ . Similarly, for any  $y \in \Sigma_+^{u,E}$  with  $|s| \leq a$  and  $|u_2| \leq 2\delta$ , we have  $C_E^{s,c}(y) \neq \emptyset$ .*

*Proof.* We note that

$$\nabla N = (\lambda_1 u_1 + u O_1, \lambda_2 u_2 + u O_1, \lambda_1 s_1 + s O_1, \lambda_2 s_2 + s O_1),$$

and hence for small  $\|u\|$ ,  $\nabla N \sim (0, 0, \lambda_1 s_1, \lambda_2 s_2)$ . Since  $|s_2| \leq 2\delta = 2|s_1|$  on  $\Sigma_+^s$ , we have the angle between  $\nabla N$  and  $u_1$  axis is bounded from below. As a consequence, there exists  $c > 0$ , such that  $C^{u,c}$  has nonempty intersection with the tangent direction of  $S_E$  (which is orthogonal to  $\nabla N$ ). The lemma follows.  $\square$

*Proof of Theorem 21.* We will apply Proposition 8.5 to the pair  $x_T^{++}$  and  $y_T^{++}$  which we will denote by  $x_T$  and  $y_T$  for short. Since the curve  $\gamma^+$  is tangent to the  $s_1$ -axis, for  $\delta$  sufficiently small, we have  $p^+ = (\delta, s_2^+, 0, 0)$  satisfies  $|s_2| \leq \delta$ . As  $x_T \rightarrow p^+$ , for sufficiently large  $T$ , we have  $x_T = (s_1, s_2, u_1, u_2)$  satisfy  $|u| \leq a/2$  and  $|s_2| \leq 3\delta/2$ , where  $a$  is as in Lemma 8.6. As a consequence, for each  $x' \in U_{a/2}(x_T) \cap \Sigma_+^{s,E}$ , we have  $C_E^{u,c}(x') \neq \emptyset$ . Similarly, we conclude that for each  $y' \in U_{a/2}(y_T) \cap \Sigma_+^{u,E}$ ,  $C_E^{s,c}(y') \neq \emptyset$ . We may choose  $T_0$  such that  $a/2 \geq \sqrt{c^2 + 1}re^{-(\lambda_2 - \kappa)T_0}$ .

Let  $\bar{\gamma}$  be a stable curve containing  $x_T$  extended to the boundary of  $B_{r/2}$ . Denote the intersection with the boundary  $\bar{x}_1$  and  $\bar{x}_2$  and let  $\bar{y}_1$  and  $\bar{y}_2$  be their images under  $\varphi_T$ . Let  $\gamma'_{13}$  and  $\gamma'_{24}$  be unstable curves containing  $\bar{y}_1$  and  $\bar{y}_2$  extended to the boundary of  $B_r$ , and let  $\gamma_{13}$  and  $\gamma_{24}$  be their preimages under  $\varphi_T$ . Pick  $x_1$  and  $x_3$  on the curve  $\gamma_{13}$  and let  $y_1$  and  $y_3$  be their images. It is possible to pick  $x_1$  and  $x_3$  such that the segment  $y_1 y_3$  on  $\gamma'_{13}$  extends beyond  $B_{r/2}$ . We now let  $\gamma_{12}$  and  $\gamma_{34}$  be stable curves containing  $x_1$  and  $x_3$  that intersects  $\gamma_{24}$  at  $x_2$  and  $x_4$ .

Note that by construction,  $\bar{\gamma}$  and  $\gamma'_{13}$  are extended to the boundary of  $B_{r/2}$ . As the parameter  $T \rightarrow \infty$ , the limit of the corresponding curves still extends to the boundary of  $B_{r/2}$ , which contains  $\gamma_s^+$  and  $\gamma_u^+$  respectively. Moreover, by Proposition 8.5,

the Hausdorff distance between  $\gamma_{12}$ ,  $\gamma_{34}$  and  $\bar{\gamma}$  is exponentially small in  $T$ , hence they have a common limit. The same can be said about  $\gamma'_{13}$  and  $\gamma'_{24}$ .

There exists a Poincaré map taking  $\gamma_{12}$  and  $\gamma_{34}$  to curves on the section  $\Sigma_+^s$ ; we abuse notation and still call them  $\gamma_{12}$  and  $\gamma_{34}$ . Similarly,  $\gamma'_{13}$  and  $\gamma'_{24}$  can also be mapped to the section  $\Sigma_+^u$  by a Poincaré map. These curves on the sections  $\Sigma_+^s$  and  $\Sigma_+^u$  completely determines the rectangle  $R^{++}(T) \subset \Sigma_+^{s,E(T)}$ . Note that the limiting properties described in the previous paragraph is unaffected by the Poincaré map. This concludes the proof of Theorem 21.  $\square$

By construction curves  $\gamma_{12}$  and  $\gamma_{34}$  can be selected as stable and  $\gamma_{14}$  and  $\gamma_{23}$  — as unstable. It leads to the following

**Corollary 8.7.** *There exists  $T_0 > 0$  such that the following hold.*

1. *For  $T \geq T_0$ ,  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}(R^{++}(T))$  intersects  $R^{++}(T)$  transversally. Moreover, the images of  $\gamma_{13}$  and  $\gamma_{24}$  intersect  $\gamma_{12}$  and  $\gamma_{34}$  transversally, and the images of  $\gamma_{12}$  and  $\gamma_{34}$  does not intersect  $R^{++}(T)$ .*
2. *For  $T \geq T_0$ ,  $\Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{--}(R^{--}(T))$  intersects  $R^{--}(T)$  transversally.*
3. *For  $T, T' \geq T_0$  such that  $R^{+-}(T)$  and  $R^{-+}(T')$  are on the same energy surface:  $\Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{+-}(R^{+-}(T))$  intersect  $R^{-+}(T')$  transversally, and  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{-+}(R^{-+}(T'))$  intersect  $R^{+-}(T)$  transversally.*

**Remark 8.2.** *Later we show that, for fixed  $T$ , the value  $T'$  satisfying condition in the third item is unique.*

In the next three sections we prove existence and uniqueness of shadowing period orbits. This results in a proof of Theorem 6.

## 8.4 Conley-McGehee isolating blocks

We will use Theorem 21 to prove Theorem 6. We apply the construction in the previous section to all four local maps in the neighborhoods of the points  $p^\pm$  and  $q^\pm$ , and obtain the corresponding rectangles.

For the map  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}|_{S_{E(T)}}$ , the rectangle  $R^{++}(T)$  is an isolating block in the sense of Conley and McGehee ([63]), defined as follows.

A rectangle  $R = I_1 \times I_2 \subset \mathbb{R}^d \times \mathbb{R}^k$ ,  $I_1 = \{\|x_1\| \leq 1\}$ ,  $I_2 = \{\|x_2\| \leq 1\}$  is called *an isolating block* for the  $C^1$  diffeomorphism  $\Phi$ , if the following hold:

1. The projection of  $\Phi(R)$  to the first component covers  $I_1$ .
2.  $\Phi|_{I_1 \times \partial I_2}$  is homotopically equivalent to the identity restricted on the set  $I_1 \times (\mathbb{R}^k \setminus \text{int } I_2)$ .

If  $R$  is an isolating block of  $\Phi$ , then the set

$$W^+ = \{x \in R : \Phi^k(x) \in R, k \geq 0\}$$

$$(\text{resp. } W^- = \{x \in R : \Phi^{-k}(x) \in R, k \geq 0\})$$

projects onto  $I_1$  (resp. onto  $I_2$ ) (see [63]). If some additional cone conditions are satisfied, then  $W^+$  and  $W^-$  are in fact  $C^1$  graphs. Note that in this case,  $W^+ \cap W^-$  is the unique fixed point of  $\Phi$  on  $R$ .

As usual, we denote by  $C^{u,c}(x) = \{c\|v_1\| \leq \|v_2\|\}$  the unstable cone at  $x$ . We denote by  $\pi C^{u,c}(x)$  the set  $x + C^{u,c}(x)$ , which corresponds to the projection of the cone  $C^{u,c}(x)$  from the tangent space to the base set. The stable cones are defined similarly. Let  $U \subset \mathbb{R}^d \times \mathbb{R}^k$  be an open set and  $\Phi : U \rightarrow \mathbb{R}^d \times \mathbb{R}^k$  a  $C^1$  map. Denote  $D\Phi_x$  the linearization of  $\Phi$  at  $x$ .

C1.  $D\Phi_x$  preserves the cone field  $C^{u,c}(x)$ , and there exists  $\Lambda > 1$  such that  $\|D\Phi(v)\| \geq \Lambda\|v\|$  for any  $v \in C^{u,c}(x)$ .

C2.  $\Phi$  preserves the projected restricted cone field  $\pi C^{u,c}$ , i.e., for any  $x \in U$ ,

$$\Phi(U \cap \pi C^{u,c}(x)) \subset C^{u,c}(\Phi(x)) \cap \Phi(U).$$

C3. If  $y \in \pi C^{u,c}(x) \cap U$ , then  $\|\Phi(y) - \Phi(x)\| \geq \Lambda\|y - x\|$ .

The unstable cone condition guarantees that any forward invariant set is contained in a Lipschitz graph.

**Proposition 8.8** (See [63]). *Assume that  $\Phi$  and  $U$  satisfies [C1]-[C3], then any forward invariant set  $W \subset U$  is contained in a Lipschitz graph over  $\mathbb{R}^k$  (the stable direction).*

Similarly, we can define the conditions [C1]-[C3] for the inverse map and the stable cone, and refer to them as “stable [C1]-[C3]” conditions. Note that if  $\Phi$  and  $U$  satisfies both the isolating block condition and the stable/unstable cone conditions, then  $W^+$  and  $W^-$  are transversal Lipschitz graphs. In particular, there exists a unique intersection, which is the unique fixed point of  $\Phi$  on  $R$ . We summarize as follows.

**Corollary 8.9.** *Assume that  $\Phi$  and  $U$  satisfies the isolating block condition, and that  $\Phi$  and  $U$  (resp.  $\Phi^{-1}$  and  $U \cap \Phi(U)$ ) satisfies the unstable (resp. stable) conditions [C1]-[C3]. Then  $\Phi$  has a unique fixed point in  $U$ .*

## 8.5 Single leaf cylinders

We now apply the isolating block construction to the maps and rectangles obtained in Corollary 8.7.

**Proposition 8.10.** *There exists  $T_0 > 0$  such that the following hold.*

- For  $T \geq T_0$ ,  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}$  has a unique fixed point  $p^+(T)$  on  $\Sigma_+^s \cap R^{++}(T)$ ;
- For  $T \geq T_0$ ,  $\Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{--}$  has a unique fixed point  $p^-(T)$  on  $\Sigma_-^s \cap R^{--}(T)$ ;
- For  $T, T' \geq T_0$  such that  $R^{+-}(T)$  and  $R^{-+}(T')$  are on the same energy surface:  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{+-} \circ \Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{-+}$  has a unique fixed point  $p^c(T)$  on  $R^{+-}(T) \cap (\Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{+-})^{-1}(R^{-+}(T'))$ .

Note that in the third case of Proposition 8.10, it is possible to choose  $T'$  depending on  $T$  such that the rectangles are on the same energy surface, if  $T$  is large enough. Moreover, as in remark 8.2 we later show that such  $T' = T'(T)$  is unique. As a consequence, the fixed point  $p^c(T)$  exists for all sufficiently large  $T$ .

Each of the fixed points  $p^+(T)$ ,  $p^-(T)$  and  $p^c(T)$  corresponds to a periodic orbit of the Hamiltonian flow. In addition, the energy of the orbits are monotone in  $T$ , and hence we can switch to  $E$  as a parameter.

**Proposition 8.11.** *The curves  $(p^+(T))_{T \geq T_0}$ ,  $(p^-(T))_{T \geq T_0}$  and  $(p^c(T))_{T \geq T_0}$  are  $C^1$  graphs over the  $u_1$  direction with uniformly bounded derivatives. Moreover, the energy  $E(p^+(T))$ ,  $E(p^-(T))$  and  $E(p^c(T))$  are monotone functions of  $T$ . In particular, with notations of section 3.4 if the family of minimal geodesics  $\{\gamma_E^{h,\pm}\}_E$  with  $0 < E < E_0$  as the limit  $\lim_{E \rightarrow 0} \gamma_{E,\pm}^h = \gamma^\pm$  has a simple loop, then for  $E_0$  small enough such a family is unique as well as the family  $\{\gamma_E^c\}_E$  with  $-E_0 < E < 0$ .*

We now prove Theorem 6 assuming Propositions 8.10 and 8.11.

*Proof of Theorem 6.* Note that due to Proposition 8.2, the sign of  $s_1$  and  $u_1$  does not change in the boundary value problem. It follows that the energies of  $p^\pm(T)$  are positive, and the energy of  $p^c(T)$  is negative. Reparametrize by energy, we obtain families of fixed points  $(p^\pm(E))_{0 < E \leq E_0}$  and  $(p^c(E))_{-E_0 \leq E < 0}$ , where

$$E_0 = \min\{E(p^+(T_0)), E(p^-(T_0)), -E(p^c(T_0))\}.$$

We now denote the full orbits of these fixed points  $\gamma_E^+$ ,  $\gamma_E^-$  and  $\gamma_E^c$ , and the theorem follows.  $\square$

To prove Proposition 8.10, we notice that the rectangle  $R^{++}(T)$  has  $C^1$  sides, and there exists a  $C^1$  change of coordinates turning it to a standard rectangle. It's easy to see that the isolating block conditions are satisfied for the following maps and rectangles:

$$\begin{aligned} & \Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++} \quad \text{and} \quad R^{++}(T), \quad \Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{--} \quad \text{and} \quad R^{--}(T), \\ & \Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{+-} \circ \Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{+-} \quad \text{and} \quad (\Phi_{\text{glob}}^- \circ \Phi_{\text{loc}}^{+-})^{-1} R^{++}(T) \cap R^{+-}(T). \end{aligned}$$

It suffices to prove the stable and unstable conditions [C1]-[C3] for the corresponding return map and rectangles. We will only prove the [C1]-[C3] conditions for the unstable cone  $C_E^{u,c}$ , the map  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}$  and the rectangle  $R^{++}(T)$ ; the proof for the other cases can be obtained by making obvious changes to the case covered.

**Lemma 8.12.** *There exists  $T_0 > 0$  and  $c > 0$  such that the following hold. Assume that  $U \subset \Sigma_+^s \cap B_r$  is a connected open set on which the local map  $\Phi_{\text{loc}}^{++}$  is defined, and for each  $x \in U$ ,*

$$\inf\{t \geq 0 : \varphi_t(x) \in \Sigma_+^u\} \geq T_0.$$

*Then the map  $D(\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++})$  preserves the non-empty cone field  $C^{u,c}$ , and the inverse  $D(\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++})^{-1}$  preserves the non-empty  $C^{s,c}$ . Moreover, the projected cones  $\pi C^{u,c} \cap U$  and  $\pi C^{s,c} \cap V$  are preserved by  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}$  and its inverse, where  $V = \Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}(U)$ .*

*The same set of conclusions hold for the restricted version. Namely, we can replace  $C^{u,c}$  and  $C^{s,c}$  with  $C_E^{u,c}$  and  $C_E^{s,c}$ , and  $U$  with  $U \cap S_E$ .*

Let  $x \in U$  and denote  $y = \Phi_{\text{loc}}^{++}(x)$ . We will first show that  $D\Phi_{\text{loc}}^{++}(x)C^{u,c}(x)$  is very close to the strong unstable direction  $T^{uu}$ . In general, we expect the unstable cone to contract and get closer to the  $T^{uu}$  direction along the flow. The limiting size of the cone depends on how close the flow is to a linear hyperbolic flow. We need the following auxiliary Lemma.

Assume that  $\varphi_t$  is a flow on  $\mathbb{R}^d \times \mathbb{R}^k$ , and  $x_t$  is a trajectory of the flow. Let  $v(t) = (v_1(t), v_2(t))$  be a solution of the variational equation, i.e.  $v(t) = D\varphi_t(x_t)v(0)$ . Denote the unstable cone  $C^{u,c} = \{\|v_1\|^2 < c\|v_2\|^2\}$ .

**Lemma 8.13.** *With the above notations assume that there exists  $b_2 > 0$ ,  $b_1 < b_2$  and  $\sigma, \delta > 0$  such that the variational equation*

$$\dot{v}(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

*satisfy  $A \leq b_1 I$  and  $D \geq b_2 I$  as quadratic forms, and  $\|B\| \leq \sigma$ ,  $\|C\| \leq \delta$ .*

*Then for any  $c > 0$  and  $\kappa > 0$ , there exists  $\delta_0 > 0$  such that if  $0 < \delta, \sigma < \delta_0$ , we have*

$$(D\varphi_t)C^{u,c} \subset C^{u,\beta_t}, \quad \beta_t = ce^{-(b_2-b_1-\kappa)t} + \sigma/(b_2 - b_1 - \kappa).$$



*Proof.* Denote  $\gamma_0 = c$ . The invariance of the cone field is equivalent to

$$\frac{d}{dt} (\beta_t^2 \langle v_2(t), v_2(t) \rangle - \langle v_1(t), v_1(t) \rangle) \geq 0.$$

Compute the derivatives using the variational equation, apply the norm bounds and the cone condition, we obtain

$$2\beta_t (\beta'_t + (b_2 - \delta\beta_t - b_1)\beta_t - \sigma) \|v_2\|^2 \geq 0.$$

We assume that  $\beta_t \leq 2\gamma_0$ , then for sufficiently small  $\delta_0$ ,  $\delta\beta_t \leq \kappa$ . Denote  $b_3 = b_2 - b_1 - \kappa$  and let  $\beta_t$  solve the differential equation

$$\beta'_t = -b_3\beta_t + \sigma.$$

It's clear that the inequality is satisfied for our choice of  $\beta_t$ . Solve the differential equation for  $\beta_t$  and the lemma follows.  $\square$

*Proof of Lemma 8.12.* We will only prove the unstable version. By Assumption 4, there exists  $c > 0$  such that  $D\Phi_{\text{glob}}^+(q^+)T^{uu}(q^+) \subset C^{u,c}(p^+)$ . Note that as  $T_0 \rightarrow \infty$ , the neighborhood  $U$  shrinks to  $p^+$  and  $V$  shrinks to  $q^+$ . Hence there exists  $\beta > 0$  and  $T_0 > 0$  such that  $D\Phi_{\text{glob}}^+(y)C^{u,\beta}(y) \subset C^{u,c}$  for all  $y \in V$ .

Let  $(s, u)(t)_{0 \leq t \leq T}$  be the trajectory from  $x$  to  $y$ . By Proposition 8.2, we have  $\|s\| \leq e^{-(\lambda_1 - \kappa)T/2}$  for all  $T/2 \leq t \leq T$ . It follows that the matrix for the variational equation

$$\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} = \begin{bmatrix} -\text{diag}\{\lambda_1, \lambda_2\} + O(s) & O(s) \\ O(u) & \text{diag}\{\lambda_1, \lambda_2\} + O(u) \end{bmatrix} \quad (28)$$

satisfies  $A \leq -(\lambda_1 - \kappa)I$ ,  $D \geq (\lambda_1 - \kappa)I$ ,  $\|C\| = O(\delta)$  and  $\|B\| = O(e^{-(\lambda_1 - \kappa)T/2})$ . As before  $C^{u,c}(x) = \{\|v_s\| \leq c\|v_u\|\}$ , Lemma 8.13 implies

$$D\varphi_T(x)C^{u,c}(x) \subset C^{u,\beta_T}(y),$$

where  $\beta_T = O(e^{-\lambda'T/2})$  and  $\lambda' = \min\{\lambda_2 - \lambda_1 - \kappa, \lambda_1 - \kappa\}$ . Finally, note that  $D\varphi_T(x)C^{u,c}(x)$  and  $D\Phi_{\text{loc}}^{++}(x)C^{u,c}(x)$  differs by the differential of the local Poincaré map near  $y$ . Since near  $y$  we have  $|s| = O(e^{-(\lambda_1 - \kappa)T})$ , using the equation of motion, the Poincaré map is exponentially close to identity on the  $(s_1, s_2)$  components, and is exponentially close to a projection to  $u_2$  on the  $(u_1, u_2)$  components. It follows that the cone  $C^{u,\beta_T}$  is mapped by the Poincaré map into a strong unstable cone with exponentially small size. In particular, for  $T \geq T_0$ , we have

$$D\Phi_{\text{loc}}^{++}(x)C^{u,c}(x) \subset C^{u,\beta}(y),$$

and the first part of the lemma follows. To prove the restricted version we follow the same arguments.  $\square$

Conditions [C1]-[C3] follows, and this concludes the proof of Proposition 8.10.

*Proof of Proposition 8.11.* Again, we will only treat the case of  $p^+(T)$ . Note that  $l^+(p^+) := (p^+(T))_{T \geq T_0}$  is a forward invariant set of  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}$ , and by Lemma 8.12, the map  $\Phi_{\text{glob}}^+ \circ \Phi_{\text{loc}}^{++}$  also preserves the (unrestricted) strong unstable cone field  $C^{u,c}$ . Apply Proposition 8.8, we obtain that  $l^+(p^+)$  is contained in a Lipschitz graph over the  $s_1 u_1 u_2$  direction. Since  $l^+(p^+)$  is also backward invariant, and using the invariance of the strong stable cone fields, we have  $l^+(p^+)$  is contained in a Lipschitz graph over the  $s_1 u_1 s_2$  direction. The intersection of the two Lipschitz graph is a Lipschitz graph over the  $s_1 u_1$  direction. Since  $l^+(p^+) \subset \{s_1 = \delta\}$ , we conclude that  $l^+(p^+)$  is Lipschitz over  $u_1$ . Since the fixed point clearly depends smoothly on  $T$ ,  $l^+(p^+)$  is a smooth curve. The Lipschitz condition ensures a uniform derivative bound. This proves the first claim of the proposition. Note that this also implies  $u_1$  is a monotone function of  $T$ .

For the monotonicity, note that all  $p^+(T)$  are solutions of the Shil'nikov boundary value problem. By definition  $(p^+(T))_{T > T_0}$  belong to  $\Sigma_+^s$  and we have  $s_1 = \delta$ . For all finite  $T$  the union of  $(p^+(T))_{T > T_0}$  is smooth. Since  $l^+(p^+)$  is a Lipschitz graph over  $u_1$  for small  $u_1$ , we have that the tangent  $(ds_2, du_1, du_2)$  is well-defined and ratios  $\frac{ds_2}{du_1}$  and  $\frac{du_2}{du_1}$  are bounded.

Theorem 20 implies that the  $s_2, u_2$  components are dominated by the  $s_1, u_1$  directions, namely, there exist  $C > 0$  and  $\alpha > 0$  such that for components of  $p^+(T)$  and all  $T > T_0$  we have  $|u_2| \leq C|u_1|^\alpha$ .

Using the form of the energy given by Corollary 8.1 its differential has the form

$$\begin{aligned} dE(s, u) &= (\lambda_1 + O(s, u)) s_1 du_1 + (\lambda_1 + O(s, u)) u_1 ds_1 + \\ &\quad + (\lambda_2 + O(s, u)) s_2 du_2 + (\lambda_2 + O(s, u)) u_2 ds_2. \end{aligned}$$

On the section  $\Sigma_+^s$  differential  $ds_1 = 0$  and coefficients in front of  $ds_2$  can be make arbitrary small. Therefore, to prove monotonicity of  $E(p^+(T))$  in  $T$  it suffices to prove that for any  $\tau > 0$  there is  $T_0 > 0$  such that for any  $T > T_0$  tangent of  $l^+(p^+)$  at  $p^+(T)$  satisfies  $|\frac{du_2}{du_1}| < \tau$ . Indeed,  $(s_1, s_2)(T) \rightarrow (\delta, s_2^*)$  as  $T \rightarrow \infty$ .

We prove this using Lemma 8.13 and the form of the equation in variations (28). Suppose  $|\frac{du_2}{du_1}| > \tau$  for some  $\tau > 0$  and arbitrary small  $u_1$ . If  $T_0$  is large enough, then  $T > T_0$  is large enough and  $u_1$  is small enough. By Theorem 20 we have  $|u_2| \leq C|u_1|^\alpha$  so  $u_2$  is also small enough. Thus, we can apply Lemma 8.13 with  $v_1 = (s_1, s_2, u_1)$  and  $v_2 = u_2$ . It implies that the image of a tangent to  $l^+(p^+)$  after application of  $D\Phi_{\text{loc}}^{++}$  is mapped into a small unstable cone  $C^{u,\beta}$  with  $\beta = (e^{-(\lambda_2 - \lambda_1 - \varepsilon)T_0} + O(\delta))/\tau$ . However, the image of  $l^+(p^+)$  under  $D\Phi_{\text{loc}}^{++}$  by definition is  $(q^+(T))_{T \geq T_0}$  and its tangent can't be in an unstable cone. This is a contradiction.

As a consequence, the energy  $E(p^+(T))$  depends monotonically on  $u_1$ . Combine with the first part, we have  $E(p^+(T))$  depends monotonically on  $T$ .  $\square$

## 8.6 Double leaf cylinders

In the case of the double leaf cylinder, there exist two rectangles  $R_1$  and  $R_2$ , whose images under  $\Phi_{\text{glob}} \circ \Phi_{\text{loc}}$  intersect themselves transversally, providing a “horseshoe” type picture.

**Proposition 8.14.** *There exists  $E_0 > 0$  such that the following hold:*

1. *For all  $0 < E \leq E_0$ , there exist rectangles  $R_1(E), R_2(E) \in \Sigma_+^{s,E}$  such that for  $i = 1, 2$ ,  $\Phi_{\text{glob}}^i \circ \Phi_{\text{loc}}^{++}(R_i)$  intersects both  $R_1(E)$  and  $R_2(E)$  transversally.*
2. *Given  $\sigma = (\sigma_1, \dots, \sigma_n)$ , there exists a unique fixed point  $p^\sigma(E)$  of*

$$\prod_{i=n}^1 (\Phi_{\text{glob}}^{\sigma_i} \circ \Phi_{\text{loc}}^{++})|_{R_{\sigma_i}(E)}$$

*on the set  $R_{\sigma_1}(E)$ .*

3. *The curve  $p^\sigma(E)$  is a  $C^1$  graph over the  $u_1$  component with uniformly bounded derivatives. Furthermore,  $p^\sigma(E)$  approaches  $p^{\sigma_1}$  and for each  $1 \leq j \leq n-1$ ,*

$$\prod_{i=j}^1 (\Phi_{\text{glob}}^{\sigma_i} \circ \Phi_{\text{loc}}^{++})(p^\sigma(E))$$

*approaches  $p^{\sigma_{j+1}}$  as  $E \rightarrow 0$ .*

In particular, with notations of section 3.3 if the family of minimal geodesics  $\{\gamma_E^h\}_E$  with  $0 < E < E_0$  as the limit  $\lim_{E \rightarrow 0} \gamma_E^h = \gamma^\pm$  has a non-simple loop, then  $\gamma_E^h \cap \Sigma_+^{s,E}$  consists of exactly  $n$  distinct points  $p^{\sigma_1}(E), p^{\sigma_2}(E), \dots, p^{\sigma_n}(E)$ .

**Remark 8.3.** *By Lemma 3.2 and assumption [A0] we have that  $\gamma_h^E$  accumulates onto the union of two simple loops, possibly with multiplicities.*

*By Lemma 3.3 there is a sequence  $\sigma = (\sigma_1, \dots, \sigma_n) \in \{1, 2\}^n$ , unique up to cyclical translation, such that  $\gamma_0^h = \gamma_0^{h_{\sigma_1}} * \dots * \gamma_0^{h_{\sigma_n}}$ .*

*By construction the geodesics  $\gamma_0^{h_1}$  and  $\gamma_0^{h_2}$  intersect  $R_1(E)$  and  $R_2(E)$  respectively. Once the sequence  $\sigma$  is fixed, there is a unique fixed point for  $p^\sigma(E)$  of the composition  $\prod_{i=n}^1 (\Phi_{\text{glob}}^{\sigma_i} \circ \Phi_{\text{loc}}^{++})|_{R_{\sigma_i}(E)}$ . Thus, for  $E_0$  small enough the family  $\{\gamma_h^E\}_E$  with  $-E_0 < E < 0$  is unique as well as the family.*

**Remark 8.4.** *The second part of Theorem 6 follows from this proposition.*

*Proof.* Let  $R^{++}(E)$  be the rectangle associated to the local map  $\Phi_{\text{loc}}^{++}$  constructed in Theorem 21, reparametrized in  $E$ . Note that for sufficiently small  $\delta$ , the curve  $\gamma_s^+$  contains both  $p^1$  and  $p^2$ , and  $\gamma_u^+$  contains both  $q^1$  and  $q^2$ .

Let  $V^1 \ni q^1$  and  $V^2 \ni q^2$  be the domains of  $\Phi_{\text{glob}}^1$  and  $\Phi_{\text{glob}}^2$ , respectively. It follows from assumption A4a' that  $\Phi_{\text{glob}}^1 \gamma_u^+ \cap V^1$  intersects  $\gamma_s^+$  transversally at  $p^i$ . By Proposition 21, for sufficiently small  $E > 0$ ,  $\Phi_{\text{glob}}^1(\Phi_{\text{loc}}^{++}(R^{++}(E)) \cap V_1)$  intersects  $R^{++}(E)$  transversally. Let  $Z^1 \subset V^1$  be a smaller neighborhood of  $q^1$ . We can truncate the rectangle  $\Phi_{\text{loc}}^{++}(R^{++}(E))$  by stable curves, and obtain a new rectangle  $R'_1(E)$  such that

$$\Phi_{\text{loc}}^{++}(R^{++}(E)) \cap Z^1 \subset R'_1(E) \subset \Phi_{\text{loc}}^{++}(R^{++}(E)) \cap V^1.$$

Denote  $R_1(E) = (\Phi_{\text{loc}}^{++})^{-1}(R'_1(E))$ . The rectangles  $R_2(E)$  and  $R'_2(E)$  are defined similarly. For  $i = 1, 2$ ,  $\Phi_{\text{glob}}^i \circ \Phi_{\text{loc}}^{++}(R_i(E))$  intersects  $R^{++}(E)$ , and hence  $R_i(E)$  transversally. This proves the first statement.

Let  $R^\sigma(E)$  denote the subset of  $R_{\sigma_1}(E)$  on which the composition

$$\prod_{i=n}^1 (\Phi_{\text{glob}}^{\sigma_i} \circ \Phi_{\text{loc}}^{++})|_{R_{\sigma_i}(E)}$$

is defined.  $R^\sigma(E)$  is still a rectangle. The composition map and the rectangle  $R^\sigma(E)$  satisfy the isolation block condition and the cone conditions. As a consequence, there exists a unique fixed point.

The proof of the  $C^1$  graph property is similar to that of Proposition 8.11.  $\square$

This completes the proof of Theorem 6.

## 8.7 Normally hyperbolic invariant cylinders the slow mechanical system

In this section we will prove Theorem 7. Let us first consider the single leaf case. We will show that the union

$$\mathcal{M} := \bigcup_{0 < E \leq E_0} \gamma_E^+ \cup \bigcup_{0 < E \leq E_0} \gamma_E^- \cup \bigcup_{-E_0 \leq E < 0} \gamma_E^{+-} \cup \gamma^+ \cup \gamma^-$$

forms a  $C^1$  manifold with boundary. Denote

$$l^+(p^+) = \{p^+(E)\}_{0 < E \leq E_0}, \quad l^+(p^-) = \{p^-(E)\}_{0 < E \leq E_0},$$

$l^+(q^+) = \Phi_{\text{loc}}^{++}(l^+(p^+))$  and  $l^+(q^-) = \Phi_{\text{loc}}^{--}(l^+(q^-))$ . Note that the superscript of  $l$  indicates positive energy instead of the signature of the homoclinics. We denote

$$l^-(p^+) = \{p^c(E)\}_{-E_0 \leq E < 0}$$

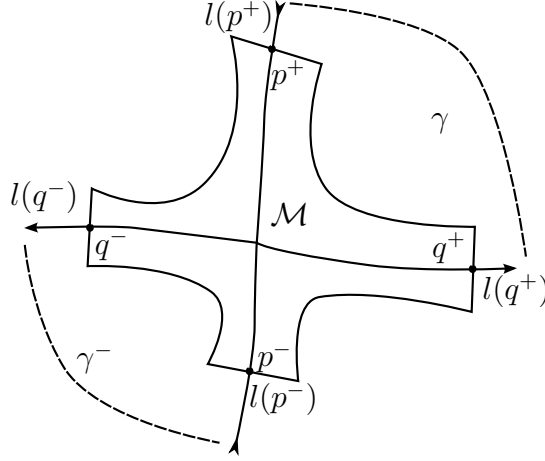


Figure 20: Invariant manifold  $\mathcal{M}$  near the origin

$l^-(q^-) = \Phi_{\text{loc}}^{+-}(l^-(p^+))$ ,  $l^-(p^-) = \Phi_{\text{glob}}^-(l^-(q^-))$  and  $l^-(q^+) = \Phi_{\text{loc}}^{-+}(l^-(p^-))$ . An illustration of  $\mathcal{M}$  the curves  $l^\pm$  are included in Figure 20.

By Proposition 8.11,  $l^\pm(y)$  ( $y$  is either  $p^\pm$ , or  $q^\pm$ ) are all  $C^1$  curves with uniformly bounded derivatives, hence they extend to  $y$  as  $C^1$  curves. Denote  $l(y) = l^+(y) \cup l^-(y) \cup \{y\}$  for  $y$  either  $p^\pm$ , or  $q^\pm$ .

**Proposition 8.15.** *There exists one dimensional subspaces  $L(p^\pm) \subset T_{p^\pm}\Sigma_\pm^u$  and  $L(q^\pm) \subset T_{q^\pm}\Sigma_\pm^s$  such that the curves  $l(p^\pm)$  are tangent to  $L(p^\pm)$  at  $p^\pm$  and  $l(q^\pm)$  are tangent to  $L(q^\pm)$  at  $q^\pm$ .*

*Proof.* Each point  $x \in l(p^+)$  contained in  $S_E$  is equal to the exiting position  $s(T_E), u(T_E)$  of a solution  $(s, u) : [0, T_E] \rightarrow B_r$  that satisfies Shil'nikov's boundary value problem (see Proposition 8.2). As  $x \rightarrow p^+$ ,  $E \rightarrow 0$  and  $T_E \rightarrow \infty$ . According to Corollary 20,  $l(p^+)$  must be tangent to the plane  $\{s_1 = u_2 = 0\}$ . Similarly,  $l(q^+)$  must be tangent to the plane  $\{u_1 = s_2 = 0\}$ . On the other hand, due to assumption A4 on the global map (see Section 3.4), the image of  $D\Phi_{\text{glob}}^+\{u_1 = s_2 = 0\}$  intersects  $\{s_1 = u_2 = 0\}$  at a one dimensional subspace. Denote this space  $L(p^+)$  and write  $L(q^+) = D(\Phi_{\text{glob}}^+)^{-1}L(p^+)$ . Since  $l(p^+)$  must be tangent to both  $\{u_2 = s_1 = 0\}$  and  $D\Phi_{\text{glob}}^+\{u_1 = s_2 = 0\}$ ,  $l(p^+)$  is tangent to  $L(p^+)$ . We also obtain the tangency of  $l(q^+)$  to  $L(q^+)$  using  $l(q^+) = (\Phi_{\text{glob}}^+)^{-1}l(p^+)$ . The case for  $l(p^-)$  and  $l(p^-)$  can be proved similarly.  $\square$

We have the following continuous version of Lemma 8.12, which states that the flow on  $\mathcal{M}$  preserves the strong stable and strong unstable cone fields. The proof of Lemma 8.16 is contained in the proof of Lemma 8.12.

**Lemma 8.16.** *There exists  $c > 0$  and  $E_0 > 0$  and continuous cone family  $C^u(x)$  and  $C^s(x)$ , such that for all  $x \in \mathcal{M}$ , the following hold:*

1.  *$C^s$  and  $C^u$  are transversal to  $T\mathcal{M}$ ,  $C^s$  is backward invariant and  $C^u$  is forward invariant.*
2. *There exists  $C > 0$  such that the following hold:*
  - $\|D\varphi_t(x)v\| \geq Ce^{(\lambda_2 - \kappa)t}$ ,  $v \in C^u(x)$ ,  $t \geq 0$ ;
  - $\|D\varphi_t(x)v\| \geq Ce^{-(\lambda_2 - \kappa)t}$ ,  $v \in C^s(x)$ ,  $t \leq 0$ .
3. *There exists a neighborhood  $U$  of  $\mathcal{M}$  on which the projected cones  $\pi C^u \cap U$  and  $\pi C^s \cap U$  are preserved.*

Note that a continuous version of Proposition 8.8 also holds. As a consequence, the set  $\mathcal{M}$  is contained in a Lipschitz graph over the  $s_1$  and  $u_1$  direction. This implies that  $\mathcal{M}$  is a  $C^1$  manifold.

**Corollary 8.17.** *The manifold  $\mathcal{M}$  is a  $C^1$  manifold with boundaries  $\gamma_{E_0}^+$ ,  $\gamma_{E_0}^-$  and  $\gamma_{-E_0}^{+-}$ .*

*Proof.* The curves  $l(p^\pm)$  and  $l(q^\pm)$  sweep out the set  $\mathcal{M} \setminus \{0\}$  under the flow. It follows that  $\mathcal{M}$  is smooth at everywhere except may be  $\{0\}$ . Since any  $x \in \mathcal{M} \cap B_r(0)$  is contained in a solution of the Shil'nikov boundary value problem, Corollary 20 implies that  $x$  is contained in the set  $\{|s_2| \leq C|s_1|^\alpha, |u_2| \leq C|u_1|^\alpha\}$ . It follows that the tangent plane of  $\mathcal{M}$  to  $x$  converges to the plane  $\{s_2 = u_2 = 0\}$  as  $(s, u) \rightarrow 0$ .  $\square$

**Corollary 8.18.** *There exists a invariant splitting  $E^s \oplus T\mathcal{M} \oplus E^u$  and  $C > 0$  such that the following hold:*

- $\|D\varphi_t(x)v\| \geq Ce^{(\lambda_2 - \kappa)t}$ ,  $v \in E^u(x)$ ,  $t \geq 0$ ;
- $\|D\varphi_t(x)v\| \geq Ce^{-(\lambda_2 - \kappa)t}$ ,  $v \in E^s(x)$ ,  $t \leq 0$ ;
- $\|D\varphi_t(x)v\| \leq Ce^{(\lambda_1 + \kappa)|t|}$ ,  $v \in T_x\mathcal{M}$ ,  $t \in \mathbb{R}$ .

*Proof.* The existence of  $E^s$  and  $E^u$ , and the expansion/contraction properties follows from standard hyperbolic arguments, see [41], for example. We now prove that third statement. Denote  $v(t) = D\varphi_t(x)v$  for  $v \in T_x\mathcal{M}$ . Decompose  $v(t)$  into  $(v_{s_1}, v_{s_2}, v_{u_1}, v_{u_2})$ , we have  $\|(v_{s_1}, v_{u_1})(t)\| \leq Ce^{(\lambda_1 + \kappa)|t|}$ . However, since  $\mathcal{M}$  is a Lipschitz graph over  $(s_1, u_1)$ , the  $(v_{s_2}, v_{u_2})$  components are bounded uniformly by the  $(v_{s_1}, v_{u_1})$  components. The norm estimate follows.  $\square$

**Remark 8.5.** *Part 1 of Theorem 7 follows from the last two corollaries.*

We now come to the double leaf case. Denote  $l(p^1) = \bigcup_{e \leq E \leq E_0} p^\sigma(E)$ , where  $p^\sigma(E)$  is the fixed point in Proposition 8.14. We have that  $l(p^{\sigma_1})$  sweeps out  $\mathcal{M}_h^{e,E_0}$  in *finite* time. As a consequence  $\mathcal{M}_h^{e,E_0}$  is a  $C^1$  manifold. Similar to Lemma 8.16, the flow on  $\mathcal{M}_h^{e,E_0}$  also preserves the strong stable/unstable cone fields. The fact that  $\mathcal{M}_h^{e,E_0}$  is normally hyperbolic follows from the invariance of the cone fields, using the same proof as that of Corollary 8.18. This concludes the proof of Theorem 7, part 2.

## 8.8 Smooth approximations

Notice that in Theorem 18 Hamiltonian  $H$  is required to be  $C^{k+1}$  with  $k \geq 9$ . In Key Theorem 3 we require  $H_\varepsilon = H_0 + \varepsilon H_1$  to be  $C^r$  with  $r \geq 4$ . Key Theorem 3 follows from Theorem 7. To fix this discrepancy we proceed with smooth approximation arguments similar to [13, Section 3.3]. More exactly, we approximate  $H_\varepsilon$  with analytic  $H_\varepsilon^*$  and prove Theorem 7 for it. Then we show that  $H_\varepsilon$  has a  $C^2$ -neighborhood of uniform size where Theorem 7 still applies. Therefore, it applies to  $H_\varepsilon$ . We need the following

**Lemma 8.19.** [69] *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^r$  function, with  $r \geq 4$ . Then for each  $\tau > 0$  there exists an analytic function  $S_\tau f$  such that*

$$\|S_\tau f - f\|_{C^s} < c(n, r) \|f\|_{C^s} \tau^{r-s} \text{ for each } 0 < s \leq r,$$

$$\|S_\tau f\|_{C^s} < c(n, r) \|f\|_{C^s} \tau^{-(s-r)} \text{ for each } s > r,$$

where  $c(n, r)$  is a constant which depends only on  $n$  and  $r$ .

Denote  $c(5, r)$  by  $c(r)$ .

We approximate  $C^r$  smooth  $H_\varepsilon = H_0 + \varepsilon H_1$  with an analytic  $H_\varepsilon^* = H_0^* + \varepsilon H_1^*$  so that  $H_0^* = S_\tau H_0$  and  $H_1^* = S_\tau H_1$  for  $\tau = \varepsilon^{2/(r-2)}$ . Apply Theorem 27 to  $H_\varepsilon^*$ . We have

$$N_\varepsilon^* = H_\varepsilon^* \circ \Phi^* = \tilde{H}_0^* + \varepsilon Z^* + \varepsilon Z_1^* + \varepsilon R^*,$$

where, in notations of Appendix B,  $Z^*(\theta, p) = [H_2^*]_{\omega_0}(\theta, p)$ ,  $\|Z_1^*(\theta, p)\| \lesssim \sqrt{\varepsilon}$ , and  $\|R^*(\theta, p, t)\| \lesssim \varepsilon$ . Consider

$$H_\varepsilon \circ \Phi^* = H_\varepsilon^* \circ \Phi^* + (H_0 - \tilde{H}_0^*) \circ \Phi^* + \varepsilon(H_1 - H_1^*) \circ \Phi^*.$$

By Lemma 8.19 applied to  $N_\varepsilon^*$  after a proper rescaling we have that it has the form

$$\tilde{H}_0^*(p_0) + \langle \partial_{pp}^2 \tilde{H}_0^*(p_0)(p - p_0), (p - p_0) \rangle + \varepsilon Z^*(\theta, p) + \varepsilon Z_1^*(\theta, p) + \varepsilon R(\theta, p, t).$$

So we can rewrite  $H_\varepsilon \circ \Phi^*$  in the form

$$\tilde{H}_0^*(p_0) + \langle \partial_{pp}^2 \tilde{H}_0^*(p_0)(p - p_0), (p - p_0) \rangle + \varepsilon Z^*(\theta, p) + \varepsilon Z_1^*(\theta, p)$$

$$+\varepsilon R(\theta, p, t) + (H_0 - H_0^*) \circ \Phi^* + \varepsilon(H_1 - H_1^*) \circ \Phi^*.$$

Recall that  $K(p - p_0)$  denotes the quadratic form  $\langle \partial_{pp}^2 H_0(p_0)(p - p_0), (p - p_0) \rangle$  up to an integer linear change of coordinates  $B$ . Denote  $K^*(p - p_0)$  the quadratic form  $\langle \partial_{pp}^2 \tilde{H}_0^*(p_0)(p - p_0), (p - p_0) \rangle$ . Define  $\|K^* - K\|$  norm of the difference between these quadratic forms by  $\max_{\|p\|=1} \|K^*(p) - K(p)\|$ . Using Lemma 8.19 we have the following estimates

- $\|H_1^* - H_1\|_{C^2} \leq c(r)\varepsilon^2,$
- $\|Z^* - Z\|_{C^2} \leq \|H_1^* - H_1\|_{C^2} \leq c(r)\varepsilon^2,$
- $\|K^* - K\| \leq \|H_0^* - H_0\|_{C^2} \leq c(r)\varepsilon^2,$
- $\|\Phi^* - Id\|_{C^2} \leq 1,$
- $\|(H_1^* - H_1) \circ \Phi^*\|_{C^2} \leq c(r)\|H_1^* - H_1\|_{C^2} \|\Phi^*\|_{C^2}^2 \leq 4c(r)\|H_1^* - H_1\|_{C^2}.$
- $\|(\tilde{H}_0^* - H_0) \circ \Phi^*\|_{C^2} \leq c(r)\|\tilde{H}_0^* - H_0\|_{C^2} \|\Phi^*\|_{C^2}^2 \leq 4c(r)\|H_0^* - H_0\|_{C^2}.$

These estimates imply that

$$\|\varepsilon R(\theta, p, t) + (H_0 - H_0^*) \circ \Phi^* + \varepsilon(H_1 - H_1^*) \circ \Phi^*\|_{C^2} \lesssim \varepsilon^2.$$

Now consider the approximating Hamiltonian  $H_\varepsilon^*$ . Using Theorem 27 and Lemma 8.19 associate a slow mechanical system  $K^*(\cdot) - Z(\cdot, p_0)$ . By Theorem 7 this Hamiltonian system has normally hyperbolic invariant manifolds  $\mathcal{M}_h^{E_0, s}$  and  $\mathcal{M}_h^{e, E_0}$  for simple critical and non-simple homologies respectively. The “moreover” part of the theorem implies persistence of these cylinders with respect to  $C^2$ -perturbations of  $H_0^*$  and  $H_1^*$ . Since the  $C^2$ -size of admissible perturbations depends only on  $\kappa$  and  $C^2$  norms of  $H^s$ , it is independent of  $\tau$ . Therefore, for small enough  $\tau$  (resp.  $\varepsilon$ ) Theorem 7 can be applied to  $H_\varepsilon = H_0 + \varepsilon H_1$ . The rest of the proof of Key Theorem 3 for  $H_\varepsilon^*$  and  $H_\varepsilon$  is the same. This completes the proof of Key Theorem 3.

## 8.9 Proof of Lemma 3.3 on cyclic concatenations of simple geodesics

We provide the proof of the auxilliary result Lemma 3.3 before proceeding to the next section.

Denote  $\gamma_1 = \gamma_0^{h_1}$  and  $\gamma_2 = \gamma_0^{h_2}$  and  $\gamma = \gamma_h^0$ . Recall that  $\gamma$  has homology class  $n_1 h_1 + n_2 h_2$  and is the concatenation of  $n_1$  copies of  $\gamma_1$  and  $n_2$  copies of  $\gamma_2$ . Since  $h_1$  and  $h_2$  generates  $H_1(\mathbb{T}^2, \mathbb{Z})$ , by introducing a linear change of coordinates, we may assume  $h_1 = (1, 0)$  and  $h_2 = (0, 1)$ .



Given  $y \in \mathbb{T}^2 \setminus \gamma \cup \gamma_1 \cup \gamma_2$ , the fundamental group of  $\mathbb{T}^2 \setminus \{y\}$  is a free group of two generators, and in particular, we can choose  $\gamma_1$  and  $\gamma_2$  as generators. (We use the same notations for the closed curves  $\gamma_i$ ,  $i = 1, 2$  and their homotopy classes). The curve  $\gamma$  determines an element

$$\gamma = \prod_{i=1}^n \gamma_{\sigma_i}^{s_i}, \quad \sigma_i \in \{1, 2\}, s_i \in \{0, 1\}$$

of this group. Moreover, the translation  $\gamma_t(\cdot) := \gamma(\cdot + t)$  of  $\gamma$  determines a new element by cyclic translation, i.e.,

$$\gamma_t = \prod_{i=1}^n \gamma_{\sigma_{i+m}}^{s_{i+m}}, \quad m \in \mathbb{Z},$$

where the sequences  $\sigma_i$  and  $s_i$  are extended periodically. We claim the following:

There exists a unique (up to translation) periodic sequence  $\sigma_i$  such that  $\gamma = \prod_{i=1}^n \gamma_{\sigma_{i+m}}$  for some  $m \in \mathbb{Z}$ , independent of the choice of  $y$ . Note that in particular, all  $s_i = 1$ .

The proof of this claim is split into two steps.

*Step 1.* Let  $\gamma_{n_1/n_2}(t) = \{\gamma(0) + (n_1/n_2, 1)t, t \in \mathbb{R}\}$ . We will show that  $\gamma$  is homotopic (along non-self-intersecting curves) to  $\gamma_{n_1/n_2}$ . To see this, we lift both curves to the universal cover with the notations  $\tilde{\gamma}$  and  $\tilde{\gamma}_{n_1/n_2}$ . Let  $p, q \in \mathbb{Z}$  be such that  $pn_1 - qn_2 = 1$  and define

$$T\tilde{\gamma}(t) = \tilde{\gamma}(t) + (p, q).$$

As  $T$  generates all integer translations of  $\tilde{\gamma}$ ,  $\gamma$  is non-self-intersecting if and only if  $T\tilde{\gamma} \cap \tilde{\gamma} = \emptyset$ . Define the homotopy  $\tilde{\gamma}_\lambda = \lambda\tilde{\gamma} + (1 - \lambda)\tilde{\gamma}_{n_1/n_2}$ , it suffices to prove  $T\tilde{\gamma}_\lambda \cap \tilde{\gamma}_\lambda = \emptyset$ . Take an additional coordinate change

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} n_1 & p \\ n_2 & q \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix},$$

then under the new coordinates  $T\tilde{\gamma}(t) = \tilde{\gamma}(t) + (1, 0)$ .

Under the new coordinates,  $T\tilde{\gamma} \cap \tilde{\gamma} = \emptyset$  if and only if any two points on the same horizontal line has distance less than 1. The same property carries over to  $\tilde{\gamma}_\lambda$  for  $0 \leq \lambda < 1$ , hence  $T\tilde{\gamma}_\lambda \cap \tilde{\gamma}_\lambda = \emptyset$ .

*Step 2.* By step 1, it suffices to prove that  $\gamma = \gamma_{n_1/n_2}$  defines unique sequences  $\sigma_i$  and  $s_i$ . Since  $\tilde{\gamma}_{n_1/n_2}$  is increasing in both coordinates, we have  $s_i = 1$  for all  $i$ . Moreover, choosing a different  $y$  is equivalent to shifting the generators  $\gamma_1$  and  $\gamma_2$ . Since the translation of the generators is homotopic to identity, the homotopy class is not affected. This concludes the proof of Lemma 3.3.

## 9 Diffusion mechanisms, weak KAM theory and forcing relation

We construct diffusing orbits using Mather variational mechanism. More specifically, Mather in his papers [53, 55, 54] proposed two different variational mechanisms of diffusion. The former was developed by Cheng-Yan [23, 24], the latter — by Bernard [9]. The foundation of Bernard’s version lies in application of weak KAM theory proposed by Fathi [34]. We rely in main part to [9] which elaborates on [54]. A convenient equivalence relation is introduced there. The advantage can be explained as follows.

When one constructs diffusion the very general idea is to find enough many invariant sets in the phase space  $\mathbb{T}^2 \times B^2 \times \mathbb{T}$ . If an invariant set is an invariant KAM torus, then one can associate rotation vector  $\omega \in B^2$ . Rotation vector can be viewed as an element of homologies  $\omega \in H_1(\mathbb{T}^2 \times \mathbb{T}, \mathbb{R})$ . In general, one could make sense of rotation vector of an invariant set with an ergodic measure on it. Then “naively” in order to diffuse it suffices to find enough many invariant sets and construct orbits going from one to the next one in turns. Certainly this vague approach faces substantial difficulties at every stage.

For convex Hamiltonian systems there is a duality between Hamiltonian dynamics and Lagrangian dynamics, between homologies and cohomologies by means of Legendre transform  $\mathbb{L}$  and Legendre-Fenchel transform  $\mathcal{L}$  respectively. Both are defined below. It turns out that a more convenient way to view invariant sets using cohomologies. Then the problem of diffusing along  $\Gamma^* \subset B^2$  in action space reduces to a problem of diffusing in velocity space along  $\mathcal{L}(\Gamma^*)$ . In [9] for two cohomologies  $c, c' \in H^1(\mathbb{T}^2, \mathbb{R})$  a forcing relation  $c \vdash c'$  is introduced. The relation  $c \dashv\vdash c'$ , defined as  $c \vdash c'$  and  $c' \vdash c$ , is an equivalence relation (Proposition 5.1 [9]). Moreover, if  $c \dashv\vdash c'$ , then there are two heteroclinic orbits going from  $c$  to  $c'$  and from  $c'$  to  $c$  (Proposition 0.10 [9]). In particular, if we can show that on  $\Gamma^*$  there is dense enough set of  $c$ ’s so that neighbors are equivalent, then end points are equivalent too. As the outcome there is an orbit going from one end point of  $\mathcal{L}(\Gamma^*)$  to the other. The approach from [53, 55, 56, 57, 23, 24] leads to a lengthy and cumbersome variational problems with constraints. Now we turn to definitions.

### 9.1 Duality of Hamiltonian and Lagrangian, homology and cohomology

Let  $M$  be a smooth compact manifold. Consider  $C^2$  Hamiltonian function  $H : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ . Denote  $(\theta, p)$  the points of  $T^*M$ . The cotangent bundle is endowed with its standard symplectic structure. We denote by  $\varphi_s^t(\theta, p)$  the time  $(t - s)$  map of the

Hamiltonian vector field of  $H$  with the initial time  $s$ , which is a time-dependent and by  $\varphi^t(\theta, p)$  the time  $t$  map with the initial time 0. Fix a Riemannian metrix  $g$  on  $M$  used to measure norms of tangent vectors and covectors on  $M$ . Assume that the following hypotheses hold.

- (*periodicity*)  $H(\theta, p, t) = H(\theta, p, t + 1)$  for each  $(\theta, p) \in T^*M$  and each  $t \in \mathbb{R}$ ;
- (*completeness*) The Hamiltonian vector field of  $H$  generates a complete flow. Namely,  $\varphi^t$  is defined for all time.
- (*convexity*) For each  $(\theta, t) \in M \times \mathbb{R}$ ,  $H(\theta, p, t)$  is strictly convex on  $T_\theta^*M$ . Namely, it has a positive definite Hessian  $\partial_{pp}^2 H$  matrix, denoted  $\partial_{pp}^2 H > 0$ .
- (*super-linearity*) For each  $(\theta, t) \in M \times \mathbb{R}$ , the function  $H(\theta, p, t)$  is super-linear in  $p$ , i.e.  $\lim_{|p| \rightarrow \infty} H(\theta, p, t)/|p| = \infty$ .

Associate to each Hamiltonian  $H(\theta, p, t)$  satisfying this list of properties a Lagrangian  $L : TM \times \mathbb{R} \longrightarrow \mathbb{R}$

$$L(\theta, v, t) = \sup_{p \in T_x^*M} v \cdot p - H(\theta, p, t),$$

where  $v \cdot p$  is the natural dot product of dual objects. This is the standard Legendre transform. Direct calculations shows that Legendre transform is involutive, i.e. Legendre transform of  $H$  is  $L$  and of  $L$  is again  $H$ . This transform gives rise to a diffeomorphism

$$\mathbb{L} : (\theta, p, t) \longrightarrow (\theta, v, t) = (\theta, \partial_p H, t),$$

whose inverse is

$$\mathbb{L}^{-1} : (\theta, v, t) \longrightarrow (\theta, p, t) = (\theta, \partial_v L, t),$$

Moreover, the Hamiltonian flow of  $H$  is mapped into the Euler-Lagrange flow of  $L$ . The Lagrangian  $L$  satisfies the following properties, which follow from the analogous properties of  $H$ :

- (*periodicity*)  $L(\theta, v, t) = L(\theta, v, t + 1)$  for each  $(\theta, v) \in TM$  and each  $t \in \mathbb{R}$ ;
- (*convexity*) For each  $(\theta, t) \in M \times \mathbb{R}$ ,  $L(\theta, v, t)$  is strictly convex on  $T_\theta M$ . Namely, it has a positive definite Hessian  $\partial_{vv}^2 L$  matrix, denoted  $\partial_{vv}^2 L > 0$ .
- (*super-linearity*) For each  $(\theta, t) \in M \times \mathbb{R}$ , the function  $L(\theta, v, t)$  is super-linear in  $v$ , which means that  $\lim_{|v| \rightarrow \infty} L(\theta, v, t)/|v| = \infty$ .

We call Lagrangians (resp. Hamiltonians) satisfying these hypothesis *Tonelli* Lagrangians (resp. *Tonelli* Hamiltonians).

In the next five sections we recall basic facts from Fathi's weak KAM theory [34] in the Hamiltonian setting. In the next chapter 10 we complement these basic facts with complementary facts stated in the (dual) Lagrangian setting. Then using this theory we introduce notion of Mather, Aubry, Mañe invariant sets and equivalence of cohomology classes proposed by Bernard [9].

## 9.2 Overlapping pseudographs

As before we let  $M$  be a smooth compact manifold and  $\pi : TM \longrightarrow M$  be the natural projection. Later we use  $M = \mathbb{T}^d$  for  $d = 2$  or  $3$ . For many invariant sets in  $TM$  we study there is a graph property. Namely,  $\mathcal{G} \subset TM$  has a graph property if it is a (usually Lipschitz) graph over  $\pi\mathcal{G} \subset M$ . The basic example are KAM tori. In this case,  $\mathcal{G}$  is a smooth graph. However, in general the invariant sets are contained in discontinuous graphs that are only forward (or backward) invariant. This leads to the definition of overlapping pseudographs.

Given  $K > 0$ , a function  $f : M \longrightarrow \mathbb{R}$  is called a  $K$ -semi-concave function if for any  $\theta \in M$ , there exists  $p_\theta \in T^*_M$ , such that for each chart  $\psi$  at  $x$ , we have

$$f \circ \psi(y) - f \circ \psi(x) \leq p_\theta(d\psi_x(y - x)) + K\|y - x\|^2$$

for all  $y$ . The linear form  $p_\theta$  is called a *super-differential* at  $\theta$ .  $f$  is called *semi-concave* if it is  $K$ -semi-concave for some  $K > 0$ . A semi-concave function is Lipschitz (see [9],(A.7)).

Given a Lipschitz function  $u : M \longrightarrow \mathbb{R}$  and a closed smooth form  $\eta$  on  $M$ , we consider the subset  $\mathcal{G}_{\eta,u}$  of  $T^*M$  defined by

$$\mathcal{G}_{\eta,u} = \{(x, \eta_x + du_x) : x \in M \text{ such that } du_x \text{ exists}\}.$$

We call the subset  $\mathcal{G} \subset T^*M$  an *overlapping pseudograph* if there exists a closed smooth form  $\eta$  and a semi-concave function  $u$  such that  $\mathcal{G} = \mathcal{G}_{\eta,u}$ . It turns out that  $\mathcal{G}$  is well suited to describe unstable manifolds. To describe stable manifolds one considers anti-overlapping pseudographs

$$\check{\mathcal{G}}_{\eta,u} = \{(x, \eta_x - du_x) : x \in M \text{ such that } du_x \text{ exists}\}.$$

Each pseudograph  $\mathcal{G}$  has a well defined cohomology class  $c = c(\mathcal{G}) \in H^1(M, \mathbb{R})$ . Indeed, if an overlapping graph is represented in two different ways as  $\mathcal{G}_{\eta,u}$  and  $\mathcal{G}_{\eta',u'}$ , the closed forms  $\eta$  and  $\eta'$  have to have the same cohomology class. Therefore, we associate to each pseudograph  $\mathcal{G}$  we associate a cohomology  $c(\mathcal{G})$  by setting

$$c(\mathcal{G}_{\eta,u}) = [\eta],$$

where  $[\eta]$  is the De Rham cohomology of  $\eta$ .

The function  $u$  is then uniquely defined up to an additive constant. As a consequence, denoting by  $\mathbb{S}$  the set of semi-concave functions on  $M$ , and by  $\mathbb{P}$  the set of overlapping pseudographs, we have the identification

$$\mathbb{P} = H^1(M, \mathbb{R}) \times \mathbb{S}/\mathbb{R}.$$

This identification endows  $\mathbb{P}$  with the structure of a real vector space. Let us endow the second factor  $\mathbb{S}/\mathbb{R}$  with the norm  $u = (\max u - \min u)/2$  which is the norm induced from the uniform norm on  $\mathbb{S}$ . More precisely, we have  $|u| = \min_v \|v\|$ , where the minimum is taken on functions  $v$  which represent the class  $u$ . We endow  $\mathbb{P}$  with the norm

$$\|\mathcal{G}_{c,u}\| = |c| + (\max u - \min u)/2,$$

The set  $\mathbb{P}$  is now a normed vector space. We define in the same way the set  $\check{\mathbb{P}}$  of anti-overlapping pseudographs  $\check{\mathcal{G}}$ , which are the sets  $\mathcal{G}_{\eta,-u}$  with  $\eta$  a smooth form and  $u \in \mathbb{S}$ . This set is similarly a normed vector space. Denote by  $\mathbb{P}_c$  the set of pseudographs with a fixed cohomology  $c \in H^1(M, \mathbb{R})$ .

### 9.3 Evolution of pseudographs and the Lax-Oleinik mapping

Denote  $C^0(M, \mathbb{R})$  the space of continuous functions on  $M$ . Let  $\Sigma(t, \theta; s, \varphi)$  be the set of absolutely continuous curves  $\gamma : [t, s] \rightarrow M$  such that  $\gamma(t) = \theta$  and  $\gamma(s) = \varphi$ . Denote by

$$d\gamma(\tau) = (\gamma(\tau), \dot{\gamma}(\tau), \tau) \quad \text{the one-jet of } \gamma(\tau).$$

It is well defined for almost every  $\tau$ . Fix a closed form  $\eta$  and a cohomology class  $c \in H^1(M, \mathbb{R})$ . Denote

$$L_c(d\gamma(\tau)) = L(d\gamma(\tau)) - c(\dot{\gamma}(\tau)) \quad L_\eta(d\gamma(\tau)) = L(d\gamma(\tau)) - \eta(\dot{\gamma}(\tau)).$$

We define the associated Lax-Oleinik mapping on  $C^0(M, \mathbb{R})$  as follows:

$$T_\eta u(\theta) = \min_{\varphi \in M, \gamma \in \Sigma(0, \varphi; 1, \theta)} \left( u(\varphi) + \int_0^1 L_\eta(d\gamma(\tau)) d\tau \right).$$

As shown by Fathi [34] (see also [9]) for each closed form  $\eta$  the functions  $T_\eta u$ ,  $u \in C(M, \mathbb{R})$  are equiv-semi-concave. Moreover, the mapping  $T_\eta$  is *contracting*:

$$\|T_\eta u - T_\eta v\|_\infty \leq \|u - v\|_\infty,$$

non-decreasing and satisfies  $T_\eta(a + u) = T_\eta u + a$  for all  $a \in \mathbb{R}$  (see e.g. [34, Corollary 4.4.4] or [9, section 2.4]).

It turns out there exists a unique mapping  $\Phi : \mathbb{P} \longrightarrow \mathbb{P}$  in the space of pseudographs

$$\Phi(\mathcal{G}_{\eta,u}) = \mathcal{G}_{u,T_\eta u}$$

for all smooth forms  $\eta$  and all semi-concave functions  $u$  (see e.g. [9, section 2.5]). We also have

$$c(\Phi(\mathcal{G})) = c(\mathcal{G}).$$

The mapping  $\Phi$  is continuous and preserves  $\mathbb{P}_c$  for each cohomology  $c \in H^1(M, \mathbb{R})$ . It turns out that the image  $\Phi(\mathbb{P}_c)$  is a relatively compact subset of  $\mathbb{P}_c$ . This follows directly from the properties of the Lax-Oleinik transformation. Along with contraction property this implies the existence of *a fixed point of  $\Phi$  in each  $\mathbb{P}_c$* , discovered by Fathi. Denote by  $\mathbb{V}_c$  the set of these fixed points, and by  $\mathbb{V} = \cup_c \mathbb{V}_c$  their union.

Even though the setting is quite abstract in the case of a twist maps one can get a feel for this transformation by drawing pictures. If  $M = \mathbb{T}$  is a circle, then overlapping pseudographs are graphs of functions which have only discontinuities with downward jumps. In other words, functions which can be locally written as the sum of a continuous and a decreasing function. Such sets were introduced by Katznelson-Ornstein [48] and many known properties of twist maps were proven. The above operator  $\Phi$  is a multidimensional generalization of this idea. To described it consider a twist map with a non-degenerate saddle fixed point. It has stable and unstable manifolds (separatrices), which have to intersect. “Trim” parts of the image of an unstable separatrix so that it is a pseudograph over  $\mathbb{T}$  and discontinuity with downward jumps only (see Figure 15). There is a freedom of choice, but it can be done so that the following fundamental property holds:

$$\overline{\Phi(\mathcal{G})} \subset \varphi^1(\mathcal{G}),$$

where as before  $\varphi = \varphi^1$  is the time one map of the Hamiltonian flow and  $c(\Phi(\mathcal{G})) = c(\mathcal{G})$ . Indeed, due to twist property the image of the pseudograph on Figure 15 at images of discontinuities will be overlapping and often inclusion is proper.

## 9.4 Aubry, Mather, Mañe sets in Hamiltonian setting and properties of forcing relation

Fathi’s weak KAM theorem states that for each cohomology class  $c \in H^1(M, \mathbb{R})$  the operator  $\Phi$  has a fixed point of cohomology  $c$  in  $\mathbb{P}_c$ . We denote the set of fixed points  $\mathbb{V}_c$ . The set of fixed points satisfy

$$\overline{\mathcal{G}} \subset \varphi(\mathcal{G}),$$

and naturally gives rise to invariant compact sets

$$\tilde{\mathcal{I}}(\mathcal{G}) := \cap_{n \geq 0} \varphi^{-n}(\mathcal{G}).$$

We shall also use notation  $\tilde{\mathcal{I}}(c, u)$ , where  $\mathcal{G} = \mathcal{G}_{\eta, u}$  and  $c = [\eta]$ . Using these invariant sets one can define another three classes of invariant sets introduced by Mather [54]. Namely, for each cohomology class  $c \in H^1(M, \mathbb{R})$  we associate the non-empty compact invariant sets

$$\tilde{\mathcal{M}}(c) \subset \tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{N}}(c),$$

where

$$\tilde{\mathcal{A}}(c) := \cap_{\mathcal{G} \in \mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G}) \quad \text{and} \quad \tilde{\mathcal{N}}(c) := \cup_{\mathcal{G} \in \mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G}), \quad (29)$$

are the Aubry set and the Mañé set. The Mather set  $\tilde{\mathcal{M}}(c)$  is the union of the supports of the invariant measures of the Hamiltonian flow  $\varphi$  on  $\tilde{\mathcal{A}}(c)$ .

In the next section we define a forcing relation introduced by Bernard [9]. The main motivation to study this relation is the following

**Proposition 9.1.** (*Proposition 0.10*) (i) Let  $\mathcal{G}$  and  $\mathcal{G}'$  be two Lagrangian graphs of cohomologies  $c$  and  $c' \in H^1(M, \mathbb{R})$ . If  $c \dashv\vdash c'$ , then there exists an integer time  $n$  such that  $\varphi^n(\mathcal{G})$  intersects  $\mathcal{G}'$ .

(ii) If  $c \dashv\vdash c'$ , there exist two heteroclinic trajectories of the Hamiltonian flow from  $\tilde{\mathcal{A}}(c)$  to  $\tilde{\mathcal{A}}(c')$  and from  $\tilde{\mathcal{A}}(c')$  to  $\tilde{\mathcal{A}}(c)$ .

(iii) Let  $\{c_i\}_{i \in \mathbb{Z}}$  be a sequence of cohomology classes all of which force the others. Fix, for each  $i$  a neighborhood  $U_i$  of the corresponding Mather  $\tilde{\mathcal{M}}(c_i)$  in  $T^*M$ . There exists a trajectory of the Hamiltonian flow  $\varphi^t$  which visits in turn all the sets  $U_i$ . In addition, if the sequence stabilizes to  $c_-$  on the left, or (and) to  $c_+$  on the right, the trajectory can be selected to be negatively asymptotic to  $\tilde{\mathcal{A}}(c_-)$  or (and) asymptotic to  $\tilde{\mathcal{A}}(c_+)$ .

The main feature is that one it is shown that in a sequence  $c_i \dashv\vdash c_{i+1}$  we have orbits traveling among the family of sets  $\{\tilde{\mathcal{M}}(c_i)\}_{i \in \mathbb{Z}}$  in any prescribed order. While the method of Mather [53, 55, 23, 24] requires a construction of a special variational problem with constraints, which often a difficult task both to write and to read. We also point out that in [25] the authors extend the result of [23, 24] from time-periodic to autonomous setting. In [26] the authors extend the result from [55] to the case when initial velocity is not assumed to be large. The preprint [55] was quite influential. Later the main result and its extensions were proved using different methods in [21, 31, 38, 42].

## 9.5 Symplectic invariance of the Mather, Aubry, and Mañe sets

The concepts of the  $\alpha$ -function, Mather, Aubry and Mañe sets are symplectic invariants. We consider the canonical one-form  $\lambda = p dx$  naturally defined on  $T^*M$ . A symplectic map

$$\Psi : T^*M \longrightarrow T^*M, \quad (X, P) \mapsto (x, p)$$

is called *exact* if the form  $\Psi^*\lambda - \lambda$  is exact. Here  $I$  and  $\theta$  are considered as functions of  $\varphi$  and  $J$ .

**Theorem 22.** [10] *Assume that  $\Psi$  is exact symplectic. Then for  $H : T^*M \longrightarrow \mathbb{R}$ , and  $c \in H^1(M, \mathbb{R})$ , we have*

$$\begin{aligned} \alpha_H(c) &= \alpha_{\Psi^*H}(\Psi^*c), & \mathcal{M}_H(0) &= \Psi(\mathcal{M}_{\Psi^*H}(\Psi^*c)), \\ \mathcal{A}_H(c) &= \Psi(\mathcal{A}_{\Psi^*H}(\Psi^*c)), & \mathcal{N}_H(0) &= \Psi(\mathcal{N}_{\Psi^*H}(\Psi^*c)). \end{aligned}$$

In particular, for the case  $M = \mathbb{T}^m$ , we can identify  $H^1(M, \mathbb{R})$  with  $\mathbb{R}^m$ . If  $\Psi$  is homotopic to identity, then  $\Psi^*\lambda = \lambda$  under this identification. Our normal form transformations are homotopic to identity, as they are constructed as time- $\epsilon$  flow of some Hamiltonian.

## 9.6 Mather's $\alpha$ and $\beta$ -functions, Legendre-Fenchel transform and barrier functions

We also need existence of so-called Mather's  $\alpha$ -function. There are several ways to introduce it. To fit to the previous discussion we use the following proposition of Fathi [34] (see Theorems 4.6.2 and 4.6.5 there).

**Proposition 9.2.** *There is a function  $\alpha : H^1(M, \mathbb{R}) \longrightarrow \mathbb{R}$  such that for each continuous function  $u$  and each closed one form  $\eta$  of cohomology class  $c$ , the sequence  $T_\eta^n u(x) + n\alpha(c)$ ,  $n \geq 1$  of continuous functions is equi-bounded and equi-Lipschitz. The function  $c \mapsto \alpha(c)$  is convex and super-linear. More precisely, there exists a constant  $K(c)$ , which does not depend on a continuous function  $u$ , such that*

$$\min u - K(c) \leq T_\eta^n u(x) + n\alpha(c) \leq \max u + K(c)$$

for each positive integer  $n$  and each point  $x \in M$ .

We also need



**Proposition 9.3.** (*Proposition 3.2 [9]*) Fix a closed one form  $\eta$  of some cohomology class  $[\eta] = c \in H^1(M, \mathbb{R})$  and a continuous function  $u$ . Set a function

$$v := \liminf_{n \rightarrow \infty} (T_\eta^n u + n\alpha(c)),$$

then  $v$  is a fixed point of  $T_\eta + \alpha(c)$  and, therefore, the corresponding pseudograph  $\mathcal{G}_{\eta, v}$  is a fixed point of  $\Phi$ .

Consider the Legendre transform of the  $\alpha$ -function:

$$\beta(h) = - \min_{c \in H^1(M, \mathbb{R})} \{\alpha(c) - c \cdot h\},$$

where  $h \in H_1(M, \mathbb{R})$  and  $c \cdot h$  is a dot product of elements of cohomologies and homologies of  $M$ . This defines a function

$$\beta : H_1(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

called *Mather's  $\beta$ -function*. Due to convexity of the  $\alpha$ -function,  $\beta$ -function is well-defined (see e.g. Theorem 20.3, [51]). Define the Legendre-Fenchel transform associated to the  $\beta$ -function.

$$\begin{aligned} \mathcal{LF}_\beta : H_1(M, \mathbb{R}) \longrightarrow \\ \text{the collection of nonempty, compact,} \\ \text{convex subsets of } H^1(M, \mathbb{R}), \end{aligned} \tag{30}$$

defined by

$$\mathcal{LF}_\beta(h) = \{c \in H^1(M, \mathbb{R}) : \beta(h) + \alpha(c) = c \cdot h\}.$$

Following Mather for any pair of points  $x, y \in M$  define a barrier function:

$$h_c(x, y) := \liminf_{n \rightarrow \infty} \int_0^n L_c(d\gamma(t)) dt + n\alpha(c),$$

where  $\gamma$  is an absolutely continuous curve with boundary conditions  $\gamma(0) = x$  and  $\gamma(n) = y$ . Notice that by Proposition 9.3 the function  $h_c(x, \cdot)$  is a fixed point of the operator  $T_c + \alpha(c)$ . Similarly, the function  $-h_c(\cdot, y)$  is a fixed point of  $\tilde{T}_c - \alpha(c)$ . Recall some basic properties of the function  $h_c$  (see e.g. Section 3.8 [9] for more details).

- For each  $x, y, z \in M$  and  $c \in H^1(M, \mathbb{R})$  there is a triangle inequality

$$h_c(x, y) + h_c(y, z) \geq h_c(x, z).$$

- For each  $x, y \in M$  and  $c \in H^1(M, \mathbb{R})$  we have  $h_c(x, y) + h_c(y, x) \geq h_c(x, x) \geq 0$ .

- For each compact set  $C \subset H^1(M, \mathbb{R})$  the set of functions  $h_c : M \times M \rightarrow \mathbb{R}$ ,  $c \in C$  is Lipschitz and even equi-semi-concave.

It is also useful to define a time-dependent barrier function on  $\mathbb{T}^2 \times \mathbb{T}$ :

$$h_c(x, t; y, s) := \liminf_{\tau - \sigma \rightarrow \infty} \int_{\sigma}^{\tau} L_c(d\gamma(t)) dt + (\tau - \sigma)\alpha(c),$$

where  $\gamma$  is an absolutely continuous curve with boundary conditions  $\gamma(\sigma) = x$  and  $\gamma(\tau) = y$ .

## 9.7 Forcing relation of cohomology classes and its dynamical properties

Following Bernard [9] section 5 we define the forcing relation  $\dashv$ , and describe its dynamical consequences. Introduce some useful notations. Given two pseudographs  $\mathcal{G}$  and  $\mathcal{G}'$  in  $TM$ , we define the relation  $\mathcal{G} \vdash \mathcal{G}'$  as follows:

$$\mathcal{G} \vdash \mathcal{G}' \iff \overline{\mathcal{G}'} \subset \bigcup_{n=1}^N \varphi^n(\mathcal{G}),$$

where as usual  $\overline{\mathcal{G}}$  is the closure of  $\mathcal{G}$ . We say that  $\mathcal{G}$  forces  $\mathcal{G}'$ , and write  $\mathcal{G} \vdash \mathcal{G}'$  if there exists an integer  $N$  such that  $\mathcal{G} \vdash_N \mathcal{G}'$ . If  $\mathcal{G}$  is a subset of  $T^*M$  and if  $c \in H^1(M, \mathbb{R})$ , the relations

$$\mathcal{G} \vdash c \quad \text{and} \quad G \vdash_N c$$

mean that there exists an overlapping pseudograph  $\mathcal{G}'$  of cohomology  $c$  and such that  $\mathcal{G} \vdash \mathcal{G}'$  (resp.  $\mathcal{G} \vdash_N \mathcal{G}'$ ). Finally, for two cohomology classes  $c$  and  $c'$ , the relation

$$c \vdash_N c'$$

means that, for each pseudograph  $\mathcal{G} \in \mathbb{P}_c$ , we have  $\mathcal{G} \vdash_N c'$ . Naturally we say that  $c$  forces  $c'$  ( $c \vdash c'$ ) if there exists an integer  $N$  such that  $c \vdash_N c'$ . The relation  $\vdash$  (between pseudographs as well as between cohomology classes) is obviously transitive. We need this relation  $\vdash$  between cohomology classes. For this purpose, it is useful to introduce the symmetric relation

$$c \dashv c' \iff c \vdash c' \quad \text{and} \quad c' \vdash c.$$

We say that  $c$  and  $c'$  force each other if  $c \dashv c'$ .

**Proposition 9.4.** (*Proposition 5.1 [9]*) *The forcing relation  $\dashv$  is an equivalence relation on  $H^1(M, \mathbb{R})$ .*

A few simple remarks about this property.

- Note that we have  $c \vdash_1 c$  for each  $c$  because by definition  $\overline{\Phi}(\mathcal{G}) \subset \varphi(\mathcal{G})$  for each  $\mathcal{G} \subset \mathbb{P}_c$ , which can be written  $\mathcal{G} \vdash_1 \Phi(\mathcal{G})$ .
- Suppose  $\mathcal{G}$  is an invariant graph (e.g. KAM torus), then the relation  $c(\mathcal{G}) \vdash c$  holds if and only if  $c = c(\mathcal{G})$ .
- Proving that two truly distinct cohomology classes  $c$  and  $c'$  with  $\tilde{\mathcal{A}}(c)$  and  $\tilde{\mathcal{A}}(c')$  disjoint have  $c \dashv\vdash c'$  is a non-trivial problem studied in [9]. In general, it does not seem sufficient to prove existence of heteroclinic orbits from  $\tilde{\mathcal{A}}(c)$  to  $\tilde{\mathcal{A}}(c')$  and back.

## 9.8 Other diffusion mechanisms and apriori unstable systems

Here we would like to give a short review of other diffusion mechanisms. In the case  $n = 2$  Arnold proposed the following example

$$H(q, p, \varphi, I, t) = \frac{I^2}{2} + \frac{p^2}{2} + \varepsilon(1 \cos q)(1 + \mu(\sin \varphi + \sin t)).$$

This example is a perturbation of the product of a one-dimensional pendulum and a one-dimensional rotator. The main feature of this example is that it has a 3-dimensional normally hyperbolic invariant cylinder. There is a rich literature on Arnold example and we do not intend to give extensive list of references; we mention [8, 14, 12, 15, 76], and references therein. This example gave rise to a family of examples of systems of  $n + 1/2$  degrees of freedom of the form

$$H_\varepsilon(q, p, \varphi, I, t) = H_0(I) + K_0(p, q) + \varepsilon H_1(q, p, \varphi, I, t),$$

where  $(q, p) \in \mathbb{T}^{n_1} \times \mathbb{R}^{n_1}$ ,  $I \in \mathbb{R}$ ,  $\varphi, t \in T$ . Moreover, the Hamiltonian  $K_0(p, q)$  has a saddle fixed point at the origin and  $K_0(0, q)$  attains its strict maximum at  $q = 0$ . For small  $\varepsilon$  a 3-dimensional NHIC  $\mathcal{C}$  persists. For  $n = 2$  systems of this type were successfully studied by different groups. Two groups were using deep geometric methods.

– In [29, 32, 39, 30] the authors carefully analyze two types of dynamics induced on the cylinder  $\mathcal{C}$ . These two dynamics are given by so-called inner and outer maps.

– In [73, 74] a return (separatrix) map along invariant manifolds of  $\mathcal{C}$  is constructed. A detailed analysis of this separatrix map gives diffusing orbits.

As we mentioned on several other occasions the other two groups [9, 23] are inspired and influenced by Mather variation method [53, 54, 55] and build diffusing orbits variationally. Recently apriori unstable structure was established for the restricted planar three body problem [35]. It turns out that for this problem there are no large gaps.

The case  $n > 2$  was studied in [24] also by a variational method. Recently Treschev [75] proved the existence of Arnold diffusion the product of a one-dimensional pendulum and  $n$ -dimensional rotator using his separatrix map approach.

A multidimensional diffusion mechanism of different nature, but also based on existence and persistence of a 3-dimensional NHIC  $\mathcal{C}$  is proposed in [19].

## 10 Properties of the barrier functions

In this section we provide several equivalent definition of some concepts introduced in Section 9. They were introduced in Hamiltonian setting. Here is concentrate on the dual description in Lagrangian setting. We will also describe properties of the barrier functions, used in our proof. There is a notational conflict. In Section 9.4 we denote by  $\tilde{\mathcal{A}}(c)$ ,  $\tilde{\mathcal{M}}(c)$ ,  $\tilde{\mathcal{N}}(c) \subset TM$  the (discrete) Aubry, Mather, Mañe sets respectively for the time 1 map  $\varphi = \varphi^1$ . In this section we denote by  $\tilde{\mathcal{A}}_H(c)$ ,  $\tilde{\mathcal{M}}_H(c)$ ,  $\tilde{\mathcal{N}}_H(c) \subset TM \times \mathbb{T}$  the (continuous) Aubry, Mather, Mañe sets respectively for the Hamiltonian flow with the Hamiltonian  $H$ . Definitions of Tonelli Hamiltonian and Tonelli Lagrangian are given in Section 9.1.

### 10.1 The Lagrangian setting

For simplicity of presentation, we restrict to the case  $M = \mathbb{T}^d$  in this section. This means we identify the spaces  $T^*M \simeq \mathbb{T}^d \times \mathbb{R}^d \simeq TM$ , and  $H^1(M, \mathbb{R}) \simeq \mathbb{R}^d$ . Given a Tonelli Hamiltonian  $H : T^*M \times \mathbb{T} \rightarrow \mathbb{R}$ , we denote by  $L_H : TM \times \mathbb{T} \rightarrow \mathbb{R}$  its associated Lagrangian.

- *The  $\alpha$ -function*

Given a cohomology  $c \in H^1(M, \mathbb{R})$ , an equivalent definition of the  $\alpha$ -function is given by

$$\alpha_H(c) := - \inf_{\mu} \int (L_H(x, v, t) - c \cdot v) d\mu(x, v, t),$$

where the infimum is taken over all invariant measures of the Euler-Lagrange flow in  $TM \times \mathbb{T}$  (see [34, Corollary 4.6.7] and [34, Theorem 4.6.2] for the equivalence with the previous definition of the  $\alpha$ -function).

- *The dual Lagrangians, minimizers, and the action function*

Given a cohomology  $c \in H^1(M, \mathbb{R})$ , denote

$$L_{H,c}(x, v, t) := L_H(x, v, t) - c \cdot v + \alpha_H(c).$$

The action function  $A_{H,c} : M \times \mathbb{R} \times M \times \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$A_{H,c}(x, \tilde{s}; y, \tilde{t}) = \inf \int_{\tilde{s}}^{\tilde{t}} L_{H,c}(\gamma(t), \dot{\gamma}(t), t) dt,$$

where the infimum is taken over all absolute continuous  $\gamma : [\tilde{s}, \tilde{t}] \rightarrow M$  such that  $\gamma(\tilde{s}) = x$ ,  $\gamma(\tilde{t}) = y$ . A curve  $\gamma$  is called a *minimizer* or a *minimizing*

*extremal* for  $A_{H,c}(x, \tilde{s}; y, \tilde{t})$  if it realizes the associated infimum. By Tonelli's theorem (see [53], Appendix 1),  $\gamma$  is always an orbit of the Euler-Lagrange flow, and hence is  $C^2$ .

- *The barrier functions*

Given a cohomology  $c \in H^1(M, \mathbb{R})$ , the barrier function  $h_{H,c} : M \times \mathbb{T} \times M \times \mathbb{T} \rightarrow \mathbb{R}$  is defined by

$$h_{H,c}(x, s; y, t) = \liminf_{n \rightarrow \infty} A_{H,c}(x, s; y, t + n),$$

where  $s$  and  $t$  are interpreted as numbers in  $[0, 1)$  on the right hand side. Up to change of notations this definition is the same as the one in section 9.6. We list it to adjust to notations of this section.

- *The projected Aubry set*

Given a cohomology  $c \in H^1(M, \mathbb{R})$ , the projected Aubry set  $\mathcal{A}_H(c)$  is defined by

$$\{(x, t) \in M \times \mathbb{T} : h_{H,c}(x, t; x, t) = 0\}.$$

We will also be using the time-zero section of the projected Aubry set, defined by

$$\mathcal{A}'(c) = \{x \in M : (x, 0) \in \mathcal{A}_H(c)\}.$$

Consider the natural projection  $\pi : TM \rightarrow M$  given by  $\pi(\theta, p, t) = (\theta, t)$ . In section 9.4 we define the Aubry set  $\tilde{\mathcal{A}}(c)$ . Denote by  $\pi\tilde{\mathcal{A}}(c) = \mathcal{A}(c)$  its projection. This definition of the Aubry set  $\mathcal{A}'(c)$  and definition (29) of the Aubry set  $\mathcal{A}(c)$  are equivalent (see e.g. Proposition 3.11 [9]), namely,  $\mathcal{A}'(c) = \mathcal{A}(c)$ .

- *The projected Mañe set*

The projected Mañe set  $\mathcal{N}_H(c)$  is defined by

$$\bigcup_{(x, t_1), (z, t_2) \in \mathcal{A}_H(c)} \{(y, t) \in M \times \mathbb{T} : h_{H,c}(x, t_1; y, t) + h_{H,c}(y, t; z, t_2) = h_{H,c}(x, t_1; z, t_2)\},$$

and the time-zero section

$$\mathcal{N}'(c) = \{x \in M : (x, 0) \in \mathcal{N}_H(c)\}.$$

In section 9.4 we define the Mañe set  $\tilde{\mathcal{N}}(c)$ . Let  $\pi\tilde{\mathcal{N}}(c) = \mathcal{N}(c)$  denote its projection. This definition of the Mañe set  $\mathcal{N}'(c)$  and definition (29) of the Mañe set  $\mathcal{N}(c)$  are equivalent (see e.g. Proposition 3.12 [9])<sup>18</sup>, namely,  $\mathcal{N}'(c) = \mathcal{N}(c)$ .

It turns out that the  $\alpha$  and  $\omega$ -limit sets of orbits of the Mañe set are contained in the corresponding Aubry set (see e.g. [11]). In this sense the Mañe set consists of heteroclinic and homoclinic orbits to “components” of Aubry set. These “components” of Aubry sets are called *static classes* and are introduced later in the section.

- *Mañe’s potential*

Following Mañe, we define the function  $m_{H,c} : M \times \mathbb{T} \times M \times \mathbb{T} \rightarrow \mathbb{R}$  by

$$m_{H,c}(x, s; y, t) = \inf_{s < t+n, n \in \mathbb{N}} A_{H,c}(x, s; y, t+n).$$

This function is sometimes called Mañe’s potential.

- *Static and semistatic curves*

Denote  $\gamma : \mathbb{R} \rightarrow M$  a  $C^1$  smooth curve and by  $d\gamma(t) = (\gamma(t), \dot{\gamma}(t), t)$  the 1-jet. A curve  $\gamma : \mathbb{R} \rightarrow M$  is called *c-semi-static* if for any  $a < b$ ,

$$\int_a^b L_{H,c}(d\gamma(t))dt = m_{H,c}(\gamma(a), a; \gamma(b), b).$$

A curve  $\gamma : \mathbb{R} \rightarrow M$  is called *static* if it is semi-static, and for all  $a < b$ ,

$$\int_a^b L_{H,c}(d\gamma(t))dt = -m_{H,c}(\gamma(b), b; \gamma(a), a).$$

By Tonelli’s Theorem, both semi-static and static curves are orbits of the Euler-Lagrange flow. Write  $p(t) = \partial_v L_H(\gamma, \dot{\gamma}, t)$ , then  $(\gamma, p)(t)$  is an orbit of the Hamiltonian flow.

- *The Aubry and Mañe sets*

Recall that in Section 9.4 we define the Aubry  $\tilde{\mathcal{A}}(c)$  and Mañe  $\tilde{\mathcal{N}}(c)$  sets in  $T^*M$ . Here we give equivalent definitions in continuous setting.

Let  $\pi : T^*(M \times \mathbb{T}) \rightarrow M \times \mathbb{T}$  be the standard projection, we have  $\pi\tilde{\mathcal{A}}(c) = \mathcal{A}(c)$  and  $\pi\tilde{\mathcal{N}}(c) = \mathcal{N}(c)$ .

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<sup>18</sup>Lemma 12.9 connection of the continuous Aubry  $\mathcal{A}_H(c)$  and Mañe  $\mathcal{N}_H(c)$  sets with the discrete ones  $\mathcal{A}'(c)$  and  $\mathcal{N}'(c)$  resp.

We define the Aubry set  $\tilde{\mathcal{A}}_H(c) \subset \mathbb{T}^*(M \times \mathbb{T})$  by

$$\{(x, p, s, -H(x, p, s)) : \varphi_s^t(x, p), t \in \mathbb{R} \text{ is static}\}.$$

By [34, Proposition 9.2.5] a curve is *static* if and only if it is a part of the projected Aubry set.

We define the Mañe set  $\tilde{\mathcal{N}}_H(c) \subset \mathbb{T}^*(M \times \mathbb{T})$  by

$$\{(x, p, s, -H(x, p, s)) : \varphi_s^t(x, p), t \in \mathbb{R} \text{ is semi-static}\}.$$

By [9, Proposition 3.6] Mañe set consists of semi-static curves.

- *Dominated functions*

A function  $u : M \times \mathbb{T} \longrightarrow \mathbb{R}$  is called *dominated* by  $L_{H,c}$  if

$$u(\gamma(t), t) - u(\gamma(s), s) \leq \int_s^t L_{H,c}(d\gamma(\sigma)) d\sigma \quad (31)$$

for each curve  $q(t) \in C^1(M, \mathbb{R})$  and each  $s < t$  in  $\mathbb{R}$ . This implies

$$u(y, t) - u(x, s) \leq h_{H,c}(x, s; y, t) \quad \forall (x, s), (y, t) \in M \times \mathbb{T}.$$

- *Calibrated curves*

A  $C^1$  curve  $\gamma : I \longrightarrow M$  is called *calibrated* on an interval  $I \subset \mathbb{R}$  by the dominated function  $u$  if, for each  $s < t$  in  $I$ , we have

$$u(\gamma(t), t) - u(\gamma(s), s) = \int_s^t L_{H,c}(d\gamma(\sigma)) d\sigma. \quad (32)$$

It is clear that calibrated curves are minimizing extremals. For each dominated function  $u$ , we define the set points in  $TM \times \mathbb{T}$

$$\begin{aligned} \tilde{I}_H(u) &= \{(x, v, t) : \\ &\quad \exists \text{ a calibrated curve } \gamma : \mathbb{R} \longrightarrow M \text{ satisfying } (x, v, t) = d\gamma(t)\}. \end{aligned} \quad (33)$$

It is known that  $\tilde{I}_H(u)$  is a compact invariant set of the Euler-Lagrange flow of  $L(H)$  (see e.g. [11]). The projection  $\pi : (x, v, t) \longrightarrow (x, t)$  induces a bi-Lipschitz homeomorphism between  $\tilde{I}_H(u)$  and its image  $I_H(u) \subset M \times \mathbb{T}$ . Fathi [34], Proposition 9.2.3 proved that a curve is semi-static if and only if it is calibrated by a dominated function  $u$ .



- *Weak KAM solutions*

The function  $u$  is called a *weak KAM solution* (of negative type) if it is dominated and if, in addition, for each point  $(x, s) \in M \times \mathbb{T}$ , there exists a calibrated curve  $\gamma : (-\infty, s) \rightarrow M$  such that  $\gamma(s) = x$ .  $u$  is a weak KAM solution of positive type if it is dominated, and for each  $(x, s)$  there is a calibrated curve  $\gamma : (s, \infty) \rightarrow M$  such that  $\gamma(s) = x$ .

Given a weak KAM solution  $u$ , we define the set

$$\overline{\mathcal{G}}_H(u) \subset TM \times \mathbb{T}$$

as points  $(x, v, s)$  such that there exists a calibrated curve  $\gamma : (-\infty, s) \rightarrow M$  satisfying  $\gamma(s) = x$  and  $\dot{\gamma}(s) = v$ . The set  $\overline{\mathcal{G}}_H(u)$  is compact and negatively invariant for the Euler-Lagrange flow  $\varphi_H^t$  of  $L_H$ . Note also that  $\pi(\overline{\mathcal{G}}_H(u)) = M \times \mathbb{T}$  and that

$$\tilde{\mathcal{I}}_H(u) = \bigcap_{t \leq 0} \varphi_H^t(\overline{\mathcal{G}}_H(u)).$$

Fathi [34] proved existence of weak KAM solutions.

- *Elementary solutions and one-sided minimizers*

For all  $(x, s) \in M \times \mathbb{T}$ , the functions  $h_{H,c}(x, s; \cdot, \cdot)$  are weak KAM solutions (see [34], Theorem 5.3.5). As a consequence, at each  $(y, t) \in M \times \mathbb{T}$ , there exists a calibrated curve

$$\gamma^- : (-\infty, t) \rightarrow M, \quad \gamma^-(t) = y.$$

We call this curve a *backward minimizer*, and it can be viewed as an extremal curve “realizing” the action  $h_{H,c}(x, s; y, t)$ . Similarly, the functions  $h_{H,c}(\cdot, \cdot; y, t)$  are positive weak KAM solutions and there exists a *forward minimizer*

$$\gamma^+ : (s, \infty) \rightarrow M \quad \text{for each} \quad (x, s) \in M \times \mathbb{T}.$$

- *Static classes*

Using the barrier function defined above we denote

$$d_H(x, s; y, t) := h_H(x, s; y, t) + h_H(y, t; x, s)$$

Mather showed that the function  $d_H$  is positive, symmetric and satisfies the triangle inequality (see e.g. [58, 59]), but there may exist points  $(x, s) \neq (y, t)$  such that  $d_H(x, s; y, t) = 0$ . One can define the relation on  $\mathcal{A}_H(c)$  by

$$(x, s) \sim (y, t) \quad \text{if} \quad d_H(x, s; y, t) = 0.$$

This is an equivalence relation on  $\mathcal{A}_H(c)$ . The equivalence classes are called the *static classes*. Their lifts to  $\tilde{\mathcal{A}}_H(c)$  are compact invariant subsets.

## 10.2 Properties of the action and barrier functions

### 10.2.1 Uniform family and Tonelli convergence

In order to state uniform properties of the minimizers over a family, we introduce the notion of uniform families of Tonelli Hamiltonians and Lagrangians (see [9]). Recall that  $\|\cdot\|_x$  denotes a norm on  $T_x M$  induced by a Riemannian metric.

A family of Tonelli Lagrangians  $\mathbb{L} \subset C^2(TM \times \mathbb{T}, \mathbb{R})$  is called *uniform* if:

1. There exist two super-linear functions  $l_0$  and  $l_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that each Lagrangian  $L$  of the family for each  $(x, t) \in M \times \mathbb{T}$  satisfies

$$l_0(\|v\|) \leq L(x, v, t) \leq l_1(\|v\|).$$

2. There exists an increasing function  $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that, if  $\varphi$  is the Euler-Lagrange flow of a Lagrangian of the family, then, for each  $t \in [-1, 1]$  and time  $t$  map  $\varphi^t$ , we have

$$\varphi^t(\{\|v\| \leq k\}) \subset \{\|v\| \leq K(k)\} \subset TM \times \mathbb{T}.$$

3. There exists a finite atlas  $\Psi$  on  $M$  such that, for each chart  $\psi \in \Psi$  and each Lagrangian  $L$  of the family, we have

$$\|\partial_{vv}^2 L \circ D\psi(q, v, t)\|_x \leq K(k) \quad \text{for} \quad \|v\| \leq k,$$

where  $D\psi$  is the differential.

We say that a family of Tonelli Hamiltonians  $\mathbb{H} \subset C^r(T^*M \times \mathbb{T}, \mathbb{R})$ ,  $r \geq 3$  is *uniform* if the family of Legendre transforms  $\mathbb{L} = \mathcal{L}(\mathbb{H})$  or, equivalently, the family of associated Lagrangians is uniform.

A sequence  $\{L_n\}_{n \geq 1}$  of Lagrangians *Tonelli converges* to  $L$  if  $\{L_n\}$  belong to a uniform family of Tonelli Lagrangians and  $L_n$  converge to  $L$  uniformly on compact sets as  $n \rightarrow \infty$ . Similarly, we say that a sequence  $\{H_n\}_{n \geq 1}$  of Tonelli Hamiltonians *Tonelli converges* to  $H$  if the sequence of corresponding Lagrangians  $L_n = L_{H_n} = \mathcal{L}(H_n)$  Tonelli converges to  $L_H = \mathcal{L}(H)$ .

Our interest in Tonelli convergence is due to the following fact. Consider a family of Hamiltonians

$$H_\varepsilon(\varphi, I, \tau) = K(I) - U(\varphi) + \sqrt{\varepsilon}P(\varphi, I, \tau), \quad (\varphi, I) \in T^*\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathbb{R}^2, \quad \tau \in \sqrt{\varepsilon}\mathbb{T},$$

where  $K(I) = \langle AI, I \rangle$  is a positive definite quadratic form,  $U$  and  $P$  are smooth functions with  $P$  being  $\sqrt{\varepsilon}$ -periodic in  $\tau$ . This is the perturbed slow system (7) derived in section B. Notice that  $H_\varepsilon$  does *not* converge to  $H_0$  in the  $C^2$ -topology, due to the fast perturbation in  $P$ . It is proved in Proposition B.5 that  $H_\varepsilon$  Tonelli converges to  $H_0$ ,

### 10.2.2 Properties of the action function

For the action function  $A_{H,c}$ , the following hold over a uniform family.

**Proposition 10.1.** *For a uniform family of Tonelli Hamiltonians  $\mathbb{H}$  the following conditions hold.*

1. (Theorem B.5, [9]) *Given a bounded set  $B \subset H^1(M, \mathbb{R})$  and  $\delta > 0$ , for each  $c \in B$  and each  $H' \in \mathbb{H}$ , there exists a uniform constant  $K > 0$  such that any minimizer  $\gamma$  of  $A_{H',c}(x, \tilde{s}; y, \tilde{t})$  with  $\tilde{t} - \tilde{s} \geq \delta$  satisfies  $\|\dot{\gamma}\| \leq K$ .*
2. (Theorem B.7, [9]) *With the same assumptions as item 1, there exists a uniform constant  $C > 0$ , such that the function  $A_{H',c}(x, \tilde{s}; y, \tilde{t})$  defined on the set  $\tilde{t} - \tilde{s} > \delta$  is  $C$ -semi-concave in  $x$  and  $y$ .*
3. (Theorem B.7, [9]) *Let  $\gamma$  be a minimizer for  $A_{H',c}(x, \tilde{s}; y, \tilde{t})$ , with  $\tilde{t} - \tilde{s} > \delta$ . Let  $p(t) = \partial_v L_H(\gamma(t), \dot{\gamma}(t))$  be the associated momentum. Then*

$$-p(\tilde{s}) + c, \quad p(\tilde{t}) - c$$

*are super-differentials of  $A_{H',c}(x, \tilde{s}; y, \tilde{t})$  at  $x$  and  $y$  respectively<sup>19</sup>.*

The action functional is also semi-concave in the time variable.

**Proposition 10.2.** *Let  $\mathbb{H} \subset C^r(T^*M \times \mathbb{T}, \mathbb{R})$ ,  $r \geq 3$  be a uniform family of Tonelli Hamiltonians. For any fixed bounded set  $B \subset H^1(M, \mathbb{R})$ ,  $K > 0$ ,  $H' \in \mathbb{H}$ , and  $c \in B$ , the function  $A_{H',c}(x, \tilde{s}; y, \tilde{t})$  is uniformly semi-concave in the  $\tilde{s}$  and  $\tilde{t}$  variables in the domain  $1 < \tilde{t} - \tilde{s} < K$ . Moreover, if  $\gamma$  is a minimizer for  $A_{H',c}(x, \tilde{s}; y, \tilde{t})$  and let  $p(t)$  be the associated momentum, then we have*

$$H'(x(\tilde{s}), p(\tilde{s}), \tilde{s}) - \alpha_{H'}(c), \quad -H'(x(\tilde{t}), p(\tilde{t}), \tilde{t}) + \alpha_{H'}(c)$$

*are super-differentials at  $\tilde{s}$  and  $\tilde{t}$  respectively.*

*Proof.* We first prove the statement for a fixed Hamiltonian  $H$  satisfying  $\alpha_H(0) = 0$ , and for  $c = 0$ . Moreover, it suffices to prove that the functions  $A_{H,0}(x, 0; y, T)$ ,  $T > 1$  and  $A_{H,0}(x, T; y, 0)$ ,  $T < -1$  are semi-concave in  $T$ . Otherwise, we will consider the Hamiltonian  $H(x, p, t + \tilde{s})$  or  $H(x, p, t + \tilde{t})$  instead.

We drop all subscripts from the notations and consider  $A(x, 0; y, T)$  for  $T > 0$ . Given any  $\Delta T \in \mathbb{R}$  with  $1 < T + \Delta T < K$ , we write  $\lambda = \Delta T/T$ . Let  $\gamma$  be a minimizer

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<sup>19</sup>In convex analysis super-differential is often called sub-differential. Since we often refer to [9] we keep his terminology to avoid further confusion

for  $A(x, 0; y, T)$ , we define  $\xi : [0, T + \Delta T] \longrightarrow M$  by  $\xi(t) = \gamma(t/(1 + \lambda))$ . We have

$$\begin{aligned} A(x, 0; y, T + \Delta T) &\leq \int_0^{T+\Delta T} L(\xi(t), \dot{\xi}(t), t) dt \\ &= \int_0^T L(\gamma(s), \dot{\gamma}(s)/(1 + \lambda), (1 + \lambda)s) ds \end{aligned}$$

with  $s = t/(1 + \lambda)$ . We will use  $O(\lambda^2)$  to denote a quantity bounded by  $C\lambda^2$ , with  $C$  depending on  $T$ ,  $\|L\|_{C^2}$ ,  $K$  and  $\sup \|\dot{\gamma}\|$ . Recall that  $d\gamma(s)$  denotes  $(\gamma(s), \dot{\gamma}(s))$ . We have

$$\begin{aligned} &A(x, 0; y, T + \Delta T) \\ &\leq (1 + \lambda) \int_0^T \left[ L(d\gamma(s)) - \partial_v L(d\gamma(s)) \cdot \frac{\lambda}{1 + \lambda} \dot{\gamma}(s) + \partial_t L(d\gamma(s)) \cdot \lambda s \right] ds + O(\lambda^2) \\ &= \int_0^T [(1 + \lambda)L(d\gamma(s)) - \lambda \partial_v L(d\gamma(s)) \cdot \dot{\gamma}(s) + \lambda \partial_t L(d\gamma(s))s] ds + O(\lambda^2). \end{aligned}$$

Hence

$$\begin{aligned} &A(x, 0; y, T + \Delta T) - A(x, 0; y, T) \\ &\leq \lambda \int_0^T (L(d\gamma(s)) - L_v(d\gamma(s)) \cdot \dot{\gamma}(s) + s \cdot \partial_t L(d\gamma(s))) ds + O(\lambda^2). \quad (34) \end{aligned}$$

We have the following calculations using the Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{ds} L(d\gamma) &= \partial_t L(d\gamma) + \partial_x L(d\gamma) \dot{\gamma} + \partial_v L(d\gamma) \ddot{\gamma} \\ &= \partial_t L(d\gamma) + \left( \partial_x L(d\gamma) - \frac{d}{ds} (\partial_v L(d\gamma)) \right) \cdot \dot{\gamma} + \frac{d}{ds} (\partial_v L(d\gamma) \cdot \dot{\gamma}) \\ &= \partial_t L(d\gamma) + \frac{d}{ds} (\partial_v L(d\gamma) \cdot \dot{\gamma}). \end{aligned}$$

Hence,

$$\partial_t L(d\gamma) = \frac{d}{ds} (L(d\gamma) - \partial_v L(d\gamma) \cdot \dot{\gamma}).$$

Using the above equality in (34), we have

$$\begin{aligned} &A(x, 0; y, T + \Delta T) - A(x, 0; y, T) \\ &\leq \lambda \int_0^T \frac{d}{ds} \left( s \cdot (L(d\gamma) - \partial_v L(d\gamma) \cdot \dot{\gamma}) \right) ds + O(\lambda^2) \\ &= \lambda T [L(d\gamma(T)) - \partial_v L(d\gamma(T)) \cdot \dot{\gamma}(T)] + O(\lambda^2) \\ &= -H(\gamma(T), p(T), T) \cdot \Delta T + O(\Delta T^2). \end{aligned}$$

This proves semi-concavity, and that in this case,  $-H(y, p(T), T)$  is a sub-differential.

For a general Hamiltonian and for  $c \neq 0$ , we reduce to the above case by considering the Hamiltonian  $H(x, p + c, t) - \alpha_H(c)$  instead. For the case of  $A(x, T; y, 0)$  with  $T < 0$ , we formally write  $A(y, 0; x, T) = A(x, T; y, 0)$ , and notice that all the above computations go through with a negative  $T$ . Finally, the constant in  $O(\cdot)$  can be chosen to be uniform for all Hamiltonians and  $c$ 's stated in the assumption. This concludes the proof for the general case.  $\square$

### 10.2.3 Properties of the barrier function

We first state some properties of semi-concave functions.

- Lemma 10.3.** *1. (Corollary A.9, [9]) Assume that  $f_n : M \rightarrow \mathbb{R}$  are  $K$ -semi-concave and  $f_n \rightarrow f$  uniformly, then  $f$  is also  $K$ -semi-concave. Moreover, let  $p_n(x)$  be super-differentials of  $f_n$  at  $x$  with  $p_n(x) \rightarrow p \in T_x M$ , then  $p$  is a super-differential of  $f$  at  $x$ .*
- 2. (Corollary A.8, [9]) Let  $f, g : M \rightarrow \mathbb{R}$  be semi-concave functions and let  $x$  be a local minimum of  $f + g$ . Then  $f$  and  $g$  are both differentiable at  $x$  with  $df(x) + dg(x) = 0$ .*

We have the following properties of the barrier function  $h_{H,c}$ .

**Proposition 10.4.** *For a uniform family of Tonelli Hamiltonians  $\mathbb{H}$  the following conditions hold.*

- 1. Given a bounded set  $B \subset H^1(M, \mathbb{R})$ , for each  $c \in B$  and each  $H \in \mathbb{H}$ , there exists a uniform constant  $C > 0$  such that the functions  $h_{H,c}(x, s; y, t)$  are  $C$ -semi-concave in  $x$  and  $y$ .*
- 2. Let a sequence  $H_n$  Tonelli converges to  $H$ ,  $c_n \rightarrow c$  and  $(x_n, s_n; y_n, t_n) \rightarrow (x, s; y, t)$ , and assume that  $h_{H_n, c_n}(x_n, s_n; \cdot, t_n) \rightarrow h_{H, c}(x, s; \cdot, t)$  uniformly. Let  $l_n$  be super-differentials of  $h_{H_n, c_n}(x_n, s_n; y, t_n)$  in  $y$  at  $y_n$  with  $l_n \rightarrow l$ , then  $l$  is a super-differential of  $h_{H, c}(x, s; y, t)$  in  $y$ . In particular, if  $h_{H_n, c_n}(x_n, s_n; y, t_n)$  is differentiable in  $y$ , then*

$$l_n \rightarrow \partial_y h_{H, c}(x, s; y, t).$$

*A similar statement holds for the super-differential in  $x$ .*

- 3. Assume that  $(x, s), (y, t) \in \mathcal{A}_H(c)$  are in the same static class. Then  $h_{H, c}(x, s; y, t)$  is differentiable in both  $x$  and  $y$ .*

*Proof.* The first two statements follow from Proposition 10.1, part 2 and Lemma 10.3, part 1.

For the third statement, we note  $h_{H,c}(x, s; \cdot, t) + h_{H,c}(\cdot, t; x, s)$  reaches its minimum value 0 at  $y$  and the conclusion follows from Lemma 10.3, part 2. The differentiability in  $(x, s)$  follow from a symmetric argument.  $\square$

**Remark 10.1.** *Geometrically, part 2 of Proposition 10.4 implies the convergence of the velocities of backward minimizers. More precisely, let  $\gamma_n$  be backward minimizers for the barrier functions  $h_{H_n,c}(x_n, s_n; y, t_n)$ ,  $\gamma$  be a backward minimizer for  $h_{H,c}(x, s; y, t)$ , and let  $p_n, p$  be the associated momentum. If  $h_{H,c}(x, s; y, t)$  is differentiable at  $y$ , then  $p(t) - c = \partial_y h_{H,c}(x, s; y, t)$  is unique. As a consequence,  $p_n(t_n) \rightarrow p(t)$  and velocities satisfy*

$$\dot{\gamma}_n(t_n) \rightarrow \dot{\gamma}(t).$$

#### 10.2.4 Semi-continuity of the Aubry and Mañe set and continuity of the barrier function

The Mañe set is semi-continuous with respect to the Hamiltonian, while the Aubry set is semi-continuous only under specific conditions.

**Proposition 10.5.** *Assume that a sequence  $H_n$  of Hamiltonians Tonelli converges to  $H$ , and  $c_n \rightarrow c \in H^1(M, \mathbb{R})$ .*

1. *(See [56], [24]) The upper limit of the projected Mañe sets  $\mathcal{N}_{H_n}(c_n)$  is contained in  $\mathcal{N}_H(c)$ . In other words, for any  $(x_n, s_n) \in \mathcal{N}_{H_n}(c_n)$ ,  $(x_n, s_n) \rightarrow (x, s)$ , we have  $(x, s) \in \mathcal{N}_H(c)$ .*
2. *(See [60], [11]) Assume in addition that the Aubry set has finitely many static class. Then the upper limit of the projected Aubry sets  $\mathcal{A}_{H_n}(c_n)$  is contained in  $\mathcal{A}_H(c)$ . In other words, for any  $(x_n, s_n) \in \mathcal{A}_{H_n}(c_n)$ ,  $(x_n, s_n) \rightarrow (x, s)$ , we have  $(x, s) \in \mathcal{A}_H(c)$ .*

In general, the barrier function  $h_{H,c}$  may be discontinuous with respect to  $H$ . However, the continuity properties hold in the particular case when the limiting Aubry set contains only one static class.

**Proposition 10.6.** *Assume that a sequence  $H_n$  of Hamiltonians Tonelli converges to  $H$ , and  $c_n \rightarrow c \in H^1(M, \mathbb{R})$ . Assume that the projected Aubry set  $\mathcal{A}_H(c)$  contains a unique static class. Let  $(x_n, s_n) \in \mathcal{A}_{H_n}(c_n)$  with  $(x_n, s_n) \rightarrow (x, s)$ , then the barrier functions  $h_{H_n,c_n}(x_n, s_n; \cdot, \cdot)$  converges to  $h_{H,c}(x, s; \cdot, \cdot)$  uniformly.*

*Similarly, for  $(y_n, t_n) \in \mathcal{A}_{H_n}(c_n)$  and  $(y_n, t_n) \rightarrow (y, t)$ , the barrier functions  $h_{H_n,c_n}(\cdot, \cdot; y_n, t_n)$  converges to  $h_{H,c}(\cdot, \cdot; y, t)$  uniformly.*

We can obtain uniformity over the choice of  $(x, s)$  in the Aubry set by using convergence up to a constant. Moreover, we can show that the super-differential of the barrier functions converges to super-differentials uniformly. In what follows, let  $\partial f(y)$  denote the set of super-differentials of a semi-concave function  $f$  at  $y$ .

**Proposition 10.7.** *Assume that a sequence  $H_n$  Tonelli converges to  $H$ ,  $c_n \rightarrow c \in H^1(M, \mathbb{R})$  and the Aubry set  $\mathcal{A}_H(c)$  contains a unique static class.*

1. *for any  $(x, s) \in \mathcal{A}_H(c)$  we have*

$$\lim_{n \rightarrow \infty} \inf_{C \in \mathbb{R}} \sup_{(y, t) \in M \times \mathbb{T}} |h_{H_n, c_n}(x_n, s_n; y, t) - h_{H, c}(x, s; y, t) - C| = 0$$

*uniformly over  $(x_n, s_n) \in \mathcal{A}_{H_n}(c_n)$  and  $(y, t) \in M \times \mathbb{T}$ .*

2. *For any  $l_n \in \partial_y h_{H_n, c_n}(x_n, s_n; y, t)$  and  $l_n \rightarrow l$ , we have  $l \in \partial_y h_{H, c}(x, s; y, t)$ . Moreover, the convergence is uniform in the sense that*

$$\lim_{n \rightarrow \infty} \inf_{l_n \in \partial_y h_{H_n, c_n}(x_n, s_n; y, t)} d(l_n, \partial_y h_{H, c}(x, s; y, t)) = 0$$

*uniformly in  $(x, s) \in \mathcal{A}_H(c)$ ,  $(x_n, s_n) \in \mathcal{A}_{H_n}(c_n)$ .*

The proofs are based on several properties of the weak KAM solutions.

**Lemma 10.8** ([34], Theorem 8.6.1, Representation formula). *Any weak KAM solution  $u(x, t)$  for  $L_{H, c}$  satisfies*

$$u(x, t) = \inf_{(x_0, t_0) \in \mathcal{A}_H(c)} \{u(x_0, t_0) + h_{H, c}(x_0, t_0; x, t)\}.$$

Using the definition of the static class, we have the following:

**Lemma 10.9.** *Assume that  $\mathcal{A}_H(c)$  has a unique static class. Let  $(x_1, t_1) \in \mathcal{A}_H(c)$ , then any weak KAM solution differs from  $h_{H, c}(x_1, t_1; \cdot, \cdot)$  by a constant.*

*Proof.* Using the definition of the static class and the triangle inequality, it is easy to see that, for any  $(x_0, t_0), (x_1, t_1) \in \mathcal{A}_H(c)$ ,

$$h_{H, c}(x_0, t_0; x, t) = h_{H, c}(x_0, t_0; x_1, t_1) + h_{H, c}(x_1, t_1; x, t).$$

Then by the representation formula,

$$\begin{aligned} u(x, t) &= \inf_{(x_0, t_0) \in \mathcal{A}_H(c)} \{u(x_0, t_0) + h_{H, c}(x_0, t_0; x_1, t_1)\} + h_{H, c}(x_1, t_1; x, t) \\ &= u(x_1, t_1) + h_{H, c}(x_1, t_1; x, t) = h_{H, c}(x_1, t_1; x, t) + \text{const.} \end{aligned}$$

□

The second statement is that weak KAM solutions are upper semi-continuous with respect to Tonelli convergence.

**Proposition 10.10** ([11], Lemma 7). *Assume that a sequence  $H_n$  Tonelli converges to  $H$  and  $c_n \rightarrow c$ , then if  $u_n$  is a weak KAM solution of  $L_{H_n, c}$  and  $u_n \rightarrow u$  uniformly, then  $u$  is a weak KAM solution of  $L_{H, c}$ .*

*Proof of Proposition 10.6.* We prove the second statement. By Proposition 10.4, all functions  $h_{H_n, c_n}(x_n, s_n; \cdot, \cdot)$  are uniformly semi-concave, and hence equi-continuous. By Arzela-Ascoli, any subsequence contains a uniformly convergent subsequence, whose limit is

$$h_{H, c}(x, s; \cdot, \cdot) + C$$

due to Proposition 10.10 and Lemma 10.9. Moreover,

$$h_{H, c}(x_n, s_n; x, s) \rightarrow h_{H, c}(x, s; x, s) = 0,$$

so  $C = 0$ . It follows that  $h_{H_n, c_n}(x_n, s_n; \cdot, \cdot)$  converges to  $h_{H, c}(x, s; \cdot, \cdot)$  uniformly.

Statement 1 follows from the definition of the projected Aubry set

$$\mathcal{A}_H(c) = \{(x, s) \in M \times \mathbb{T} : h_{H, c}(x, s; x, s) = 0\}$$

and statement 2. □

*Proof of Proposition 10.7. Part 1.* We argue by contradiction. Assume that there exist  $\delta > 0$ , and by restricting to a subsequence,

$$\inf_{C \in \mathbb{R}} \sup_{(y, t)} |h_{H_n, c_n}(x_n, s_n; y, t) - h_{H, c}(x, s; y, t) - C| > \delta.$$

By compactness, and by restricting to a subsequence again, we may assume that  $(x_n, s_n) \rightarrow (x^*, s^*)$ ,  $(y_n, t_n) \rightarrow (y, t)$ . Using Proposition 10.6, take limit as  $n \rightarrow \infty$ , we have

$$\inf_{C \in \mathbb{R}} \sup_{(y, t)} |h_{H, c}(x^*, s^*; y, t) - h_{H, c}(x, s; y, t) - C| > \delta.$$

By Lemma 10.9, the left hand side is 0, which is a contradiction.

*Part 2.* There exists constants  $C_n$  such that  $h_{H_n, c_n}(x_n, s_n; \cdot, \cdot) + C_n$  converges to  $h_{H, c}(x, s; \cdot, \cdot)$  uniformly. Convergence of super-differentials follows directly from Proposition 10.4. It suffices to prove uniformity. Assume, by contradiction, that by restricting to a subsequence, we have  $(x_n, s_n) \rightarrow (x, s) \in \mathcal{A}_H(c)$ ,  $l_n \in \partial_y h_{H_n, c_n}(x_n, s_n; y, t)$  and  $(x, s) \in \mathcal{A}_H(c)$  such that

$$\lim_{n \rightarrow \infty} l_n \notin \partial_y h_{H, c}(x, s; y, t).$$

By Proposition 10.4,  $l_n \rightarrow l \in \partial_y h_{H, c}(x^*, s^*; y, t)$ , but we also have  $\partial_y h_{H, c}(x, s; y, t) = \partial_y h_{H, c}(x^*, s^*; y, t)$  since the functions differ by a constant Lemma 10.9. This is a contradiction. □



## 11 Diffusion along the same homology at double resonance

In this section we prove Key Theorems 6 (localization of Aubry and Mañe sets), 7 (graph theorem) and 9 (forcing relation) along the same homology class.

The proof of Key Theorems 6 and 7 is covered in section 11.1, and uses well known ideas (see for example [23], [24], [10]) involving normally hyperbolic invariant cylinders.

The proof of Key Theorem 7 is more involved, and occupies section 11.2 and beyond.

### 11.1 Localization and graph theorem

We first prove Theorem 8, which is the analog of Key Theorem 6 for the system

$$H_\epsilon^s = H^s + \sqrt{\epsilon}P.$$

*Proof of Theorem 8.* The proof of all cases follows from a general argument. Assume that the Aubry set  $\mathcal{A}_{H^s}(c)$  is contained in either one or two NHICs of  $H^s$ . Assume in addition that the Aubry set is upper semi-continuous in  $H$  and  $c$  at  $(H^s, c)$ . Then there exists  $\delta, \epsilon_0 > 0$ , such that for  $|c - c'| \leq \delta$  and  $0 < \epsilon \leq \epsilon_0$ , the perturbed Aubry set  $\mathcal{A}_{H_\epsilon^s}(c')$  is contained in a small neighborhood of the unperturbed NHIC. Since the perturbed NHIC (for  $H_\epsilon^s$ ) has the weak invariance property, it contains all the invariant set in a neighborhood. This implies that the perturbed Aubry set is contained in the perturbed NHICs.

In all the cases, the unperturbed Aubry set is contained in the union of NHICs, by Proposition 4.1 for high energies, and Theorem 7 for low energies. In all cases, the Aubry set has either one or two static class, hence by Proposition 10.5, the Aubry set is upper semi-continuous. Moreover,  $\delta$  and  $\epsilon_0$  can be chosen to be uniform over all chosen cohomology classes due to compactness. All cases of Theorem 8 follows from this argument.  $\square$

Key Theorem 6 follows from Theorem 8, by the symplectic invariance of the Aubry and Mañe set.

*Proof of Key Theorem 7.* Using symplectic invariance, it suffices to prove the theorem for  $H_\epsilon^s$ .

Let  $\mathcal{M}_h^{E_* - \delta, E_* + \delta}$  be a NHIC for the slow system  $H^s$  in the homology  $h$ , for energy  $[E_* - \delta, E_* + \delta]$ . The NHIC consists of periodic orbits  $\gamma_h^E$ , and let  $s \in \mathbb{T}$  be a parametrization of for  $\gamma_h^{E_*}$ . The pair  $(s, E)$  then defines a local coordinate system for  $\mathcal{M}_h^{E_* - \delta, E_* + \delta}$ . We now extend this coordinate system to a tubular neighborhood

of  $\mathcal{M}_h^{E_*-\delta, E_*+\delta}$ . First we let  $x$  denote the normal direction to the projection of  $\gamma_h^E$  to  $\mathbb{T}^2$ , and let  $y$  be a normal direction to  $\mathcal{M}_h^{E_*-\delta, E_*+\delta}$  that complements  $(s, E, x)$ .

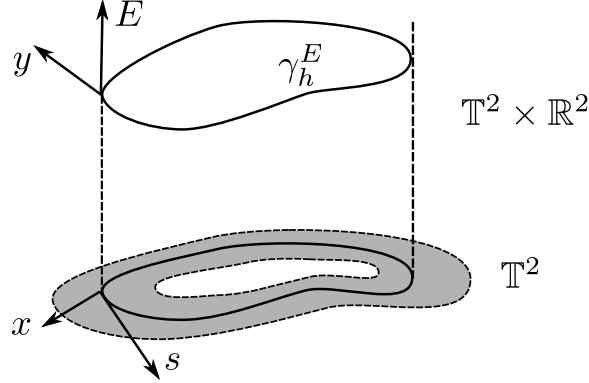


Figure 21: Local coordinates for NHIC at double resonance

The perturbed NHIC  $\mathcal{M}_{h,\epsilon}^{E_*-\delta, E_*+\delta}$  is a smooth graph over  $\mathcal{M}_h^{E_*-\delta, E_*+\delta}$ , and hence can be described as the graph  $\{(x, y) = \chi^\epsilon(s, E)\}$ . Moreover, since  $\mathcal{M}_{h,\epsilon}^{E_*-\delta, E_*+\delta}$  converges to  $\mathcal{M}_h^{E_*-\delta, E_*+\delta}$  as  $\epsilon \rightarrow 0$  in  $C^1$ -norm, we may assume  $\|\chi^\epsilon\|_{C^1} \rightarrow 0$ .

Let  $(\varphi_1, I_1)$  and  $(\varphi_2, I_2)$  be two points on  $\mathcal{A}_{H_\epsilon^s}(c_h(E)) \cap \mathcal{M}_{h,\epsilon}^{E_*-\delta, E_*+\delta}$ . Note that if the Aubry set is contained in the union of two NHICs, we consider only the points on the same cylinder. Mather's graph theorem states that the Aubry set is a Lipschitz graph over the angular variable, namely

$$\|I_1 - I_2\| \leq C\|\varphi_1 - \varphi_2\|.$$

We will use  $C$  as an unspecified generic constant. Let  $(s_1, E_1, x_1, y_1)$  and  $(s_2, E_2, x_2, y_2)$  be the coordinates of the same points, using  $E = H^s(\varphi, I)$ , we have

$$|E_1 - E_2| \leq C\|\varphi_1 - \varphi_2\| \leq C|s_1 - s_2| + C|x_1 - x_2|.$$

on the Aubry set, where the last estimate is due to  $\varphi$  being a smooth function of  $(s, x)$ . Using  $\{(x, y) = \chi^\epsilon(s, E)\}$  and  $\|\chi^\epsilon\|_{C^1} \rightarrow 0$ , we may assume for small  $\epsilon$ ,

$$|x_1 - x_2| + |y_1 - y_2| \leq \frac{1}{2C}(|E_1 - E_2| + |s_1 - s_2|),$$

and hence

$$|E_1 - E_2| \leq C|s_1 - s_2| + C\frac{1}{2C}(|E_1 - E_2| + |s_1 - s_2|) = \frac{3C}{2}|s_1 - s_2| + \frac{1}{2}|E_1 - E_2|.$$

As a consequence  $|E_1 - E_2| \leq 3C|s_1 - s_2|$ . By replacing  $C$  with a bigger constant, we obtain

$$|x_1 - x_2| + |y_1 - y_2| + |E_1 - E_2| \leq C|s_1 - s_2|.$$

This implies the projection of the Aubry set to  $s$  coordinate has an Lipschitz inverse.

We note that the above argument applies whenever  $\gamma_h^E$  is a smooth closed curve, this covers all energy  $E \neq 0$  and for a simple non-critical  $h$ ,  $E = 0$  as well.

The only remaining case is  $E = 0$  for a simple critical  $h$ . In this case  $\gamma_h^0$  is not smooth in  $\mathbb{T}^2 \times \mathbb{R}^2$ , its tangent vector is discontinuous at the hyperbolic fixed point  $o$ . However, its projection to  $\mathbb{T}^2$  is smooth. Our local coordinates  $(s, x)$  are well defined. The function  $E$  is not  $C^1$  at  $o$ , however, it is Lipschitz. However, we may modify  $E$  near the corner point to a  $C^1$  function, while keeping the Lipschitz property. This provides a well defined local coordinate system, and the above argument applies.  $\square$

## 11.2 Forcing relation along the same homology class

The rest of this section is dedicated to proofs of Key Theorems 8 and 9 which are very similar. Most of the section is devoted to the latter one as it involves several different cases. As we wrote in section 6.2 the proof of Key Theorem 9 (resp. Key Theorem 8) divides into two steps:

- perturb so that all cohomologies under consideration are of at most three types (see Theorems 13 and 11 resp.);
- perturb again so that all cohomologies under consideration belong to a single forcing class (see Theorems 14 and 12).

Before we start proving these two Theorems notice that in Key Theorem 9 we have the following regimes:

- (high energy)  $h$  is simple and  $E_0 \leq E \leq \bar{E}$ ;
- (high energy)  $h$  is non-simple and  $e \leq E \leq \bar{E}$  for some  $0 < e < E_0$ .
- (low energy)  $h_1$  and  $h'_1$  are simple, critical and  $0 \leq E \leq 2E_0$ ;
- (low energy)  $h$  is simple, non-critical and  $0 \leq E \leq E_0$ .

In Key Theorem 8 we have only two regimes:

- the corresponding Aubry sets are localized in one cylinder;

- (bifurcation case) the corresponding Aubry sets are localized inside of two cylinders and have non empty intersection with both;

It corresponds to the following families of cohomologies:  $\Gamma_h^{E_0, \bar{E}}$ ,  $\Gamma_{h,f}^{e, \bar{E}}$ ,  $\Gamma_{\pm h_1, s}^{0, 2E_0}$  and  $\Gamma_{\pm h'_1, s}^{0, 2E_0}$ ,  $\Gamma_{\pm h}^{0, E_0}$  respectively.

It turns out the cases simple, high energy and non-simple, high energy are similar. The cases simple, critical low energy and simple, non-critical low energy are also similar. An additional subtlety in the case of two simple critical homologies  $h_1$  and  $h'_1$  is that we need localized perturbations in Theorem 13 with disjoint supports.

We start discussing a simple homology, high energy case. Then describe how it applies to non-simple homology, high energy. After that we move to the case of two simple critical homologies, low energy followed by modifications needed to apply arguments to simple non-critical homologies, low energy.

Recall that by Key Theorems 6 and 7 (by Key Theorems 4 + 5 resp.) the corresponding Aubry sets are localized inside of the corresponding cylinders<sup>20</sup> and satisfy Mather's projected graph property respectively.

Recall that for a mechanical system  $H^s(\theta^s, I^s) = K(I^s) - U(\theta^s)$ , energy  $E > 0$ , and an integer homology class  $h \in H^1(\mathbb{T}^s, \mathbb{Z})$  we define a cohomology class  $\bar{c}_h(E)$ . This is the homology class whose Fenichel-Legendre transform is  $h$  (see Proposition 4.1 for the definition). By Proposition B.4 in a double resonance after a canonical change of coordinates and proper rescaling the Hamiltonian has the form

$$H_\varepsilon^s(\theta^s, I^s, \tau) = K(I^s) - U(\theta^s) + \sqrt{\varepsilon} P(\theta^s, I^s, \tau),$$

$$\theta^s \in \mathbb{T}^s \simeq \mathbb{T}^2, \quad I^s \in \mathbb{R}^2, \quad \tau \in \sqrt{\varepsilon} \mathbb{T}.$$

The next Theorem is an improvement of Theorem 13 and relies on Key Theorems 6 and 7. Key Theorem 6 proves that, in notations of section 4.2.3, the family of Aubry sets  $\{\tilde{\mathcal{A}}_{H_\varepsilon^s}(\bar{c}_h(E))\}_{E \in [E_0, \bar{E}]}$  and  $\{\tilde{\mathcal{A}}_{H_\varepsilon^s}(\lambda \bar{c}_h(0))\}_{0 \leq \lambda < 1}$  are localized in the corresponding cylinders  $\mathcal{M}_{h, \varepsilon}^{E', E''}$  with  $E \in [E', E'']$  and  $E'$  and  $E''$  properly chosen<sup>21</sup>. Key Theorem 7 shows also that each of these Aubry sets are graphs over the corresponding geodesics  $\gamma_h^E$ 's. The Theorem below insures that there are only finitely many  $E'$ 's with  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\bar{c}_h(E))$  having nonempty components inside two distinct cylinders. Moreover, each nonempty component  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\bar{c}_h(E)) \cap \mathcal{M}_{h, \varepsilon}^{E', E''}$  contains a unique minimal invariant probability measure. Here the precise statement.

**Theorem 23.** *In the setting and notations of Theorem 13 there exists an arbitrarily small  $C^r$  perturbation  $\sqrt{\varepsilon} \bar{P}$  of  $\sqrt{\varepsilon} P$  such that the perturbation  $\sqrt{\varepsilon}(\bar{P} - P)$  is localized*

<sup>20</sup>For simple non-critical low energy we also need to add the hyperbolic periodic orbit  $o_\varepsilon$  at zero energy (see Key Theorem 6, item 4)

<sup>21</sup>In the case of simple non-critical we need to add the hyperbolic periodic orbit  $o_\varepsilon$

near the normally hyperbolic weakly invariant cylinders in listed Theorem 13 and the Hamiltonian  $\bar{H}_\varepsilon^s$  satisfies the following conditions:

1. Finiteness of bifurcations

- (a) (high energy) If  $h$  is simple and satisfies the conditions [DR1]-[DR3]. Then there exists a partition of  $[E_0, \bar{E}]$  into  $\bigcup_{j=0}^{l-1} [\bar{E}_j, \bar{E}_{j+1}]$ , which is a refinement of the partition  $\{[E_i, E_{i+1}]\}$ , each  $[\bar{E}_j, \bar{E}_{j+1}] \subset [E_i, E_{i+1}]$  for some  $i$ , with the property that items  $i$  and  $ii$  of the previous case hold.
  - i. for  $E \in (\bar{E}_j, \bar{E}_{j+1})$ , the Aubry set  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(c_h(E))$  is contained in a normally hyperbolic weakly invariant manifold  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$ ;
  - ii. for  $E = \bar{E}_{j+1}$ ,  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\bar{c}_h(E))$  has nonempty component in both cylinders  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$  and  $\mathcal{M}_{h,\varepsilon}^{\bar{E}_{j+1}, E_{j+2}}$ .
- (b) (high energy) If  $h$  is non-simple and satisfies the conditions [DR1]-[DR3]. Then there exists a partition of  $[E_0, \bar{E}]$  into  $\bigcup_{j=0}^{l-1} [\bar{E}_j, \bar{E}_{j+1}]$ , which is a refinement of the partition  $\{[E_i, E_{i+1}]\}$ , each  $[\bar{E}_j, \bar{E}_{j+1}] \subset [E_i, E_{i+1}]$  for some  $i$ , with the property that items  $i$  and  $ii$  of the previous case hold.
- (c) (low energy) If  $h$  be simple non-critical and satisfies the conditions [DR1]-[DR3]. Then there exists a partition of  $[-E_0, E_0]$  into  $\bigcup_{j=0}^{l-1} [\bar{E}_j, \bar{E}_{j+1}]$  with the property that
  - i. for  $E \in (\bar{E}_j, \bar{E}_{j+1})$ , the Aubry set  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\bar{c}_h(E))$  is contained either in one of normally hyperbolic weakly invariant manifolds  $\mathcal{M}_{\pm h, \varepsilon}^{0, E_0}$  or in  $o_\varepsilon$ ;
  - ii. for  $E = \bar{E}_{j+1}$ ,  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\bar{c}_h(E))$  has nonempty component in two out of three sets given above.

2. No invariant curve of minimal homoclinic orbits to the origin

(low energy) If  $h_1$  and  $h'_1$  are simple, critical homologies and satisfy the conditions [DR1]-[DR3] and conditions [A1]-[A4] of Key Theorem 3, by Key Theorem 6, item 3 for each  $0 \leq \lambda \leq 1$  the family of Aubry sets  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\lambda \bar{c}_h(0)) \subset \tilde{\mathcal{N}}_{\bar{H}_\varepsilon^s}(\lambda \bar{c}_h(0))$  is localized in a normally hyperbolic weakly invariant manifold  $\mathcal{M}_{h,\varepsilon}^{E_0, s}$  and

$$\lambda^* = \min\{\lambda \in (0, 1) : (\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\lambda \bar{c}_h(0)) \setminus o_\varepsilon) \cap \mathcal{M}_{h,\varepsilon}^{E_0, s} \neq \emptyset\}. \quad (35)$$

By Key Theorem 7, item 3  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\lambda \bar{c}_h(0))$  is a graph over  $\gamma_h^0$ . Then  $\tilde{\mathcal{A}}_{\bar{H}_\varepsilon^s}(\lambda^* \bar{c}_h(0)) \cap \{t = 0\}$  is a discrete non-empty set and  $\lambda^* \bar{c}_h(0) \in \Gamma_{h,s}^{0, E_0}$ .

In all four case we also have

### 3. Ergodicity of Aubry sets

- (a) The sets  $\tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E)) \cap \mathcal{M}_{h,\epsilon}^{E_i, E_{i+1}}$  (resp.  $\mathcal{M}_{h,\epsilon}^{0, E_0}$  or  $\mathcal{M}_{h,\epsilon}^{E_0, s}$ ), when nonempty, contains a unique minimal invariant probability measure. In particular, this implies that  $\tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E)) = \tilde{\mathcal{N}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E))$  for  $E \neq \bar{E}_j$  for any  $j$ .
- (b) An immediate consequence of part (a) is the following dichotomy, for  $E \neq \bar{E}_j$ ,  $j = 1, \dots, l$ , one of the two holds.
  - i.  $\tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E)) = \tilde{\mathcal{N}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E))$  and  $\pi_{\gamma_h^E} \tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E)) = \gamma_h^E$ . In this case,  $\tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E))$  is an invariant circle.
  - ii.  $\pi_{\gamma_h^E} \tilde{\mathcal{N}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E)) \subsetneq \gamma_h^E$ , where  $\pi_{\gamma_h^E}$  is a composition of the natural projection  $\pi : \mathbb{T}^s \times \mathbb{R}^2 \times \sqrt{\epsilon} \mathbb{T} \longrightarrow \mathbb{T}^s$  and the projection onto  $\gamma_h^E$  along normals to  $\gamma_h^E$ <sup>22</sup>.

## 11.3 Local extensions of the Aubry sets and a proof of Theorem 23

Proof of Theorem 23 is similar to the proof of Theorem 6.1 [13]. In this section we prove item 1 (finiteness of bifurcations). Consider first the case  $h$  is either simple, high energy or non-simple, high energy. The case 1, (c) turns out to be similar.

By Key Theorem 6, item 1 for each energy  $E \in [E_0, \bar{E}]$  which is not  $\delta$ -close to any bifurcation value  $\{E_i\}_{i=1, \dots, N-1}$  we have  $E \in [E_j + \delta, E_{j+1} - \delta]$  for some  $j = 1, \dots, N-1$  and that the corresponding Aubry set  $\tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E))$  is localized inside only one normally hyperbolic weakly invariant cylinder  $\mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}$ . The family of Aubry sets  $\tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E))$  as a function of  $E$  has “jumps” (or bifurcations) from one cylinder to another one. These bifurcations can occur only for  $E \in [E_j - \delta, E_j + \delta]$  by Key Theorem 6, item 2. In order to understand the bifurcations we extend

$$\tilde{\mathcal{A}}_{\tilde{H}_\epsilon^s}(\bar{c}_h(E))|_{E \in [E_j + b, E_{j+1} - b]}$$

from  $[E_j + b, E_{j+1} - b]$  to  $[E_j - 2b, E_{j+1} + 2b]$ . By definition the cylinder  $\mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}$  extends to energies  $[E_j - \delta, E_{j+1} + \delta] \supset [E_j - 2b, E_{j+1} + 2b]$ . It turns out one can make sense of an extended local Aubry set, which is localized inside  $\mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}$ . This definition is inspired by Mather’s definition of a relative  $\alpha$ -function and a relative Aubry set (see also section 6.1 [13]).

Recall that for each energy there are at most two minimal geodesics of the Jacobi metric  $\rho_E$  on  $\mathbb{T}^s \simeq \mathbb{T}^2$ . For a finitely many energies  $E \in \{E_j\}_{j=1}^N$  there are exactly two

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<sup>22</sup>It is well defined, because by Theorem 3 the projection  $\pi \tilde{\mathcal{N}}_{\bar{c}_h(E)}$  is contained in a tube neighborhood of  $\gamma_h^E$ . This projection is well-defined in a tube neighborhood

minimal geodesics  $\gamma_E^h$  and  $\bar{\gamma}_E^h$ . By the condition [DR2] for  $E = E_j$  both geodesics can be smoothly continued to the interval  $[E_j - \delta, E_j + \delta] \ni E$  and denoted  $\gamma_E^h$  and  $\bar{\gamma}_E^h$ . To assign dependence on  $j$  we denote by  $\gamma_{E,j}^h$  the continuation of the minimal geodesic in  $(E_j, E_{j+1})$  and by  $\gamma_{E,j-1}^h$  the continuation of the minimal geodesic in  $(E_{j-1}, E_j)$ . Let

$$\rho_0 = \max_{E \in [E_j - 2b, E_j + 2b]} \frac{\text{dist}\{\gamma_{E,j}^h, \gamma_{E,j-1}^h\}}{6},$$

Let  $b > 0$  be small enough to have

$$\max_{E, E' \in [E_j - 2b, E_j + 2b]} \{\text{dist}\{\gamma_{E,j}^h, \gamma_{E',j}^h\}, \text{dist}\{\gamma_{E,j-1}^h, \gamma_{E',j-1}^h\}\} \leq \frac{\rho_0}{2}.$$

For small  $\rho > 0$  we denote

$$T_i(\rho) = \{\theta^s \in \mathbb{T}^s : \text{dist}(\theta^s, \gamma_{E_j,i}^h) < \rho\}, \quad i = j-1, j$$

tube neighborhoods of  $\gamma_{E_j,j-1}^h$  and  $\gamma_{E_j,j}^h$  respectively. By definition  $T_{j-1}(\rho_0)$  and  $T_j(\rho_0)$  are disjoint.

Consider two smooth extensions of averaged potentials  $U$ , denoted  $U_j$  and  $U_{j-1}$  respectively and defined as follows:

$$\begin{aligned} U_j(\theta^s) &= \begin{cases} U(\theta^s), & \text{if } \theta^s \in T_j(\rho_0/2) \\ \geq U(\theta^s) + \frac{b}{2} \text{dist}(\theta^s, \gamma_{E_j,j}^h)^2, & \text{if } \theta^s \notin T_j(\rho_0). \end{cases} \\ U_{j-1}(\theta^s) &= \begin{cases} U(\theta^s), & \text{if } \theta^s \in T_{j-1}(\rho_0/2) \\ \geq U(\theta^s) + \frac{b}{2} \text{dist}(\theta^s, \gamma_{E_j,j-1}^h)^2, & \text{if } \theta^s \notin T_{j-1}(\rho_0), \end{cases} \end{aligned} \quad (36)$$

In the regions  $\theta^s \in T_j(\rho_0) \setminus T_j(\rho_0/2)$  (resp.  $\theta^s \in T_{j-1}(\rho_0) \setminus T_{j-1}(\rho_0/2)$ ) we smoothly interpolate so that  $U_j$  (resp.  $U_{j-1}$ ) are monotonically increasing with respect to the distance to  $\gamma_{E_j,j}^h$  (resp.  $\gamma_{E_j,j-1}^h$ ). Since  $\gamma_{E_j,j}^h$  and  $\gamma_{E_j,j-1}^h$  are smooth curve in  $\mathbb{T}^s$ , by choosing  $b$  small enough this is possible. In particular,  $U_j \geq U$  in the region  $T_j(\rho_0/2)$  (resp.  $U_{j-1} \geq U$  in the region  $T_{j-1}(\rho_0/2)$ ).

We write

$$H_{\varepsilon,i}^s(\theta^s, I^s, \tau) = K(I^s) - U_i(\theta^s) + \sqrt{\varepsilon}P(\theta^s, I^s, \tau) \quad \text{for } i = j-1, j.$$

For each  $E \in [E_j - 2b, E_{j+1} + 2b]$  we define

$$\alpha_i(\bar{c}_h(E)) = \alpha_{H_{\varepsilon,i}^s}(\bar{c}_h(E)), \quad \tilde{\mathcal{A}}_i(\bar{c}_h(E)) = \tilde{\mathcal{A}}_{H_{\varepsilon,i}^s}(\bar{c}_h(E)) \quad \text{for } i = j-1, j.$$

We need to justify that these definitions are independent of the choice of the modification  $U_i$ 's or the decompositions  $K - U_i + \sqrt{\varepsilon}P$ ,  $i = j-1, j$ . So we provide the following proposition. Recall that  $\pi : T^*\mathbb{T}^s \longrightarrow \mathbb{T}^s$  denotes the natural projection.

**Proposition 11.1.** *Let  $H_{\epsilon,j}^s(\theta^s, I^s, \tau) = K(I^s) - U_j(\theta^s) + \sqrt{\epsilon}P(\theta^s, I^s, \tau)$  be a Hamiltonian satisfying the genericity conditions [DR1]-[DR3] for an integer homology class  $h \in H_1(\mathbb{T}^s, \mathbb{Z})$ . There exists  $\epsilon_0 = \epsilon_0(K, U, E) > 0$  such that for  $0 < \epsilon < \epsilon_0$  the following hold.*

1. *The definitions of  $\alpha_j$  and  $\tilde{\mathcal{A}}_j(\bar{c}_h(E))$  are independent of the decomposition*

$$H_{\epsilon,j}^s(\theta^s, I^s, \tau) = K(I^s) - U_j(\theta^s) + \sqrt{\epsilon}P(\theta^s, I^s, \tau)$$

*as long as  $H_{0,j}^s$  satisfies [DR1]-[DR3] for  $h$ ; the definitions are also independent of the modification  $U_j$ , as long as the extensions satisfies the above properties.*

2. *For each  $E \in [E_j - 2b, E_{j+1} + 2b]$ , we have the projected local Aubry set  $\tilde{\mathcal{A}}_j(\bar{c}_h(E))$  is contained in the tube neighborhood  $T_j(\rho_0)$  by Key Theorems 2 and 6.*

*By Key Theorem 7 for Aubry sets  $\tilde{\mathcal{A}}_j(\bar{c}_h(E)) \subset \mathcal{M}_{h,\epsilon}^{E_j, E_{j+1}}$  we have that the projection  $\pi_{\theta^f}|_{\tilde{\mathcal{A}}_j(\bar{c}_h(E))}$  is one-to-one with Lipschitz inverse.*

3. *For each  $E \in [E_j - 2b, E_{j+1} + 2b]$  we have*

$$\alpha(\bar{c}_h(E)) = \max\{\alpha_j(\bar{c}_h(E)), \alpha_{j-1}(\bar{c}_h(E))\}.$$

*In particular,  $\alpha_j(\bar{c}_h(E)) > \alpha_{j-1}(\bar{c}_h(E))$  for  $E = E_j - b$  and  $\alpha_{j+1}(\bar{c}_h(E)) > \alpha_j(\bar{c}_h(E))$  for  $E = E_j + b$ .*

4. *For any  $E \in [E_j - b, E_{j+1} + b]$ , if  $\alpha(\bar{c}_h(E)) = \alpha_j(\bar{c}_h(E))$  and  $\alpha(\bar{c}_h(E)) \neq \alpha_{j-1}(\bar{c}_h(E))$ , then  $\tilde{\mathcal{A}}_{H_\epsilon^s}(\bar{c}_h(E)) = \tilde{\mathcal{A}}_j(\bar{c}_h(E))$ . Similar statement hold with  $j$  and  $j - 1$  exchanged.*

The proof uses the following lemma, which implies independence of the local Aubry set on the decomposition or the choice of the modification.

**Lemma 11.2.** *Let  $H_{\epsilon,j}^{s,'} = K - U'_j + \sqrt{\epsilon}P'$  and  $H_{\epsilon,j}^s = K - U_j + \sqrt{\epsilon}P$  be such that*

- $H_{\epsilon,j}^{s,'} = H_{\epsilon,j}^s$  for each  $\theta^s \in T_j(\rho_0)$ .
- For  $E \in [E_j - 2b, E_j + 2b]$ , we have that  $U$  and  $U_j$  satisfy conditions (36).
- $\|P\|_{C^2}, \|P'\|_{C^2} \leq 1$ .

*Then for sufficiently small  $\epsilon$  and for  $E \in [E_j - 2b, E_{j+1} + 2b]$*

$$\tilde{\mathcal{A}}_{H_{\epsilon,j}^{s,'}}(\bar{c}_h(E)) = \tilde{\mathcal{A}}_{H_{\epsilon,j}^s}(\bar{c}_h(E)).$$

The proof follows the same analysis as in the single peak case (section 5.2.1 [13]).

This completes the proof of Theorem 23 items 1.(a) and 1.(b). In the case simple, noncritical with low energy the construction is the same with replacing the pair of disjoint geodesics  $\gamma_{E,j}^h$  and  $\gamma_{E,j-1}^h$  by the other disjoint pair  $\gamma_E^h$  and  $\gamma_E^{-h}$ .



## 11.4 Generic property of the Aubry sets $\tilde{\mathcal{A}}_{H_\varepsilon^s}(\bar{c}_h(E))$

In this section we discuss the property of the sets  $\tilde{\mathcal{A}}_{H_\varepsilon^s}(\bar{c}_h(E))$  for  $E \in [E_j - 2b, E_{j+1} + 2b]$  if we allowed to make an additional perturbation. We prove Theorem 23 items 2 and 3 on absence of invariant curves of homoclinics and ergodicity of Aubry sets. It is convenient for us to fix a modified Hamiltonian  $H_{\varepsilon,j}^s$  and base all discussions on this system.

From Proposition 11.1, we have that the sets  $\tilde{\mathcal{A}}_{H_{\varepsilon,j}^s}(\bar{c}_h(E))$  (we will write  $\tilde{\mathcal{A}}_j(\bar{c}_h(E))$  for brevity in this section) are contained in the NHIC  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$ , and  $\pi_{\theta f}|_{\tilde{\mathcal{A}}_j(\bar{c}_h(E))}$  is one-to-one. We will study finer structures of the Aubry sets, by relating to the Aubry-Mather theory of two dimensional area preserving twist maps. We will prove the following statement.

**Proposition 11.3.** *There exists  $\epsilon_0 > 0$  such that for  $0 < \epsilon < \epsilon_0$ , there exists arbitrarily small  $C^r$  perturbation  $H_{\varepsilon'}^{s'}$  of  $H_\varepsilon^s$ , such that  $(H_{\varepsilon'}^{s'} - H_\varepsilon^s)$  is supported near  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$  and for each  $E \in [E_j - 2b, E_{j+1} + 2b]$ ,  $\tilde{\mathcal{A}}_{H_{\varepsilon'}^{s'}}(\bar{c}_h(E))$  supports a unique  $c$ -minimal measure.*

*Similarly, there exists an arbitrarily small  $C^r$  perturbation  $H_{\varepsilon'}^{s'}$  of  $H_\varepsilon^s$ , such that  $(H_{\varepsilon'}^{s'} - H_\varepsilon^s)$  is supported near  $\mathcal{M}_{h,\varepsilon}^{E_0, s}$  and such that for  $\lambda^*$  defined in (35) the corresponding Aubry set  $\tilde{\mathcal{A}}_{H_{\varepsilon'}^{s'}}(\lambda^* \bar{c}_h(0)) \cap \{t = 0\}$  is a discrete non-empty set and  $\lambda^* \bar{c}_h(0) \in \Gamma_{h,s}^{0, E_0}$ .*

Notice that this Proposition follows from a basic fact from dynamical systems: a generic Hamiltonian system has Kupka-Smale property, which is all periodic orbits are hyperbolic (eigenvalues are either real or not a root of unity) and stable/unstable manifolds intersect transversally (see Robinson [68]). In our case a slight difference is that we need to perturb an ambient system of 2.5 degrees of freedom and recover generic properties of a restriction onto 3-dimensional cylinder. This does not cause serious difficulties. For additional details one can see Proposition 6.6 [13].

## 11.5 Generic property of the $\alpha$ -function and proof of Theorem 13

After obtaining the desired properties for the local Aubry set, we now return to the Hamiltonian  $H_\varepsilon^s$ . If  $E \in [E_j + b, E_{j+1} - b]$ , we have that  $\tilde{\mathcal{A}}_{H_\varepsilon^s}(\bar{c}_h(E)) = \tilde{\mathcal{A}}_{H_{\varepsilon,j}^s}(\bar{c}_h(E))$ . For  $E \in [E_{j+1} - b, E_{j+1} + b]$ , Proposition 11.1, statement 3 and 4 shows that it suffices to identify whether  $\alpha(\bar{c}_h(E))$  is equal to  $\alpha_j(\bar{c}_h(E))$  or  $\alpha_{j+1}(\bar{c}_h(E))$ .

**Proposition 11.4.** *Assume that  $H_\varepsilon^s = K - U + \sqrt{\varepsilon}P$  is such that it satisfies [DR1]-[DR3] for some integer homology class  $h$ . Then there exists  $\epsilon_0 > 0$  such that for  $0 < \epsilon < \epsilon_0$ , there exists an arbitrarily small perturbation  $H_{\varepsilon'}^{s'}$  of  $H_\varepsilon^s$  such that  $(H_{\varepsilon'}^{s'} - H_\varepsilon^s)$*

is supported near  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$  and has the following properties. For the Hamiltonian  $H_\varepsilon^{s, \prime}$  Proposition 11.1 and Proposition 11.3 still hold, in addition, there exists only finitely many  $E \in [E_{j+1} - b, E_{j+1} + b]$  such that  $\alpha_j(\bar{c}_h(E)) = \alpha_{j+1}(\bar{c}_h(E))$ .

The proof relies on the following simple remark. The function  $\alpha_j(\bar{c}_h(E))$  is as a function of  $E$  is  $C^1$  for the same reason as it is  $C^1$  in the single peak case. It is easy to perturb a potential  $U_j$  so that in the tube neighborhood  $T_j(\rho_0/2)$  it is shifted by a constant  $s$  and smoothly interpolated to zero outside. Denote the family of these potentials  $U_j^s$ . Notice that the  $\alpha$ -functions  $\alpha_{j-1}$  and  $\alpha_j^s$  associated to  $c_h(E)$  associated to the Hamiltonian  $K - U_j^s + \sqrt{\varepsilon}P$  are

$$\alpha_{j-1}(c_h(E)) \quad \text{and} \quad \alpha_j^s(c_h(E)) = \alpha_j(c_h(E)) + s,$$

where  $\alpha_{j-1}(c_h(E))$  is the  $\alpha$ -function associated to  $K - U_{j-1} + \sqrt{\varepsilon}P$ . By Sard's theorem, there exists an arbitrary small regular value  $s^*$  of the difference  $\alpha_{j-1}(c_h(E)) - \alpha_j(c_h(E))$ . If  $s = s^*$  is such a value, then 0 is a regular value of  $\alpha_{j-1}(c_h(E)) - \alpha_j^s(c_h(E))$ . Therefore, there are only finitely many solutions  $E_j$  to  $\alpha_{j-1}(c_h(E)) = \alpha_j^s(c_h(E))$ . This justifies Proposition 11.4 and completes the proof of Theorem 13 in the case of simple or non-simple, high energy.

In the case of simple, non-critical, low energy the same arguments applies. The only bifurcation value of the average mechanical system is  $E = 0$ . Then we can replace  $\gamma_{E,j-1}^h$  and  $\gamma_{E,j}^h$  with  $\gamma_0^h$  and  $\gamma_0^{-h}$  and localize our perturbations near the latter ones.

Recall that  $\{\bar{E}_j\}_j \subset [E_0, \bar{E}]$  denotes a finite ordered set of bifurcation values with  $\alpha_j(\bar{c}_h(E)) = \alpha_{j+1}(\bar{c}_h(E))$  given by Proposition 11.4.

## 11.6 Nondegeneracy of the barrier functions

In this section we prove Theorem 14 and complete a proof of Key Theorem 9. We have concluded that it suffices to show that  $\Gamma_2(\epsilon) = \Gamma_2^*(\epsilon)$  and  $\Gamma_3(\epsilon) = \Gamma_3^*(\epsilon)$ , given by Theorem 14. We show that this is the case by proving the following equivalent statement.

**Proposition 11.5.** *Let  $H'_\epsilon$  be a  $C^r$  perturbation of  $H_\epsilon$  such that the conclusions of Theorem 14 holds, then there exists an arbitrarily small  $C^r$  perturbation  $H''_\epsilon$  to  $H'_\epsilon$  such that  $(H''_\epsilon - H'_\epsilon)$  is supported away from  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$  and such that for the Hamiltonian  $H''_\epsilon$  Theorem 14 still hold and we have the following properties*

1. For each  $E \in [E_0, \bar{E}]$  we have  $\mathcal{A}_{H'_\epsilon}(\bar{c}_h(E)) = \mathcal{A}_{H''_\epsilon}(\bar{c}_h(E))$ ;
2. Consider  $E \in (\bar{E}_j, \bar{E}_{j+1})$  such that  $\mathcal{A}_{H''_\epsilon}(\bar{c}_h(E)) = \mathcal{N}_{H''_\epsilon}(\bar{c}_h(E))$  and the projection  $\pi_{\theta f} [\mathcal{A}_{H''_\epsilon}(\bar{c}_h(E)) \cap \{t = 0\}] = \mathbb{T}$ . Take  $\theta \in \mathcal{A}_{H''_\epsilon}(\bar{c}_h(E))$ , and let  $\tilde{\theta}_0$  and  $\tilde{\theta}_1$

be its lifts to the double cover. We have that the set of minima of both functions

$$\begin{aligned} h_{H''_\varepsilon, \bar{c}_h(E)}(\tilde{\theta}_1, \theta) + h_{H''_\varepsilon, \bar{c}_h(E)}(\theta, \tilde{\theta}_2) \\ \text{and} \\ h_{H''_\varepsilon, \bar{c}_h(E)}(\tilde{\theta}_2, \theta) + h_{H''_\varepsilon, \bar{c}_h(E)}(\theta, \tilde{\theta}_1) \end{aligned} \tag{37}$$

outside of neighborhoods of the lifts of  $\mathcal{A}_{H''_\varepsilon}(\bar{c}_h(E))$  is totally disconnected. In other words,  $[\mathcal{N}_{H''_\varepsilon}(\bar{c}_h(E)) \setminus \mathcal{A}_{H''_\varepsilon}(\bar{c}_h(E))] \cap \{t = 0\}$  is not empty and is totally disconnected.

3. For  $E = \bar{E}_{j+1}$ , take  $\theta' \in \mathcal{A}_{H''_\varepsilon}(\bar{c}_h(E)) \cap \mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$  and  $\theta'' \in \mathcal{A}_{H''_\varepsilon}(\bar{c}_h(E)) \cap \mathcal{M}_{h,\varepsilon}^{E_{j-1}, E_j}$ . We have that the set of minima of both functions

$$\begin{aligned} h_{H''_\varepsilon, \bar{c}_h(E)}(\theta', \theta) + h_{H''_\varepsilon, \bar{c}_h(E)}(\theta, \theta'') \\ \text{and} \\ h_{H''_\varepsilon, \bar{c}_h(E)}(\theta'', \theta) + h_{H''_\varepsilon, \bar{c}_h(E)}(\theta, \theta') \end{aligned} \tag{38}$$

outside of a neighborhood of  $\mathcal{A}_{H''_\varepsilon}(\bar{c}_h(E))$  is totally disconnected. In other words,  $[\mathcal{N}_{H''_\varepsilon}(\bar{c}_h(E)) \setminus \mathcal{A}_{H''_\varepsilon}(\bar{c}_h(E))] \cap \{t = 0\}$  is not empty and totally disconnected.

**Remark 11.1.** Since properties of Aubry and Mañe sets are symplectic invariants [10], it suffices to prove this Proposition using the graph property of these cylinders in the normal form. See Key Theorem 1.

This Proposition is essentially proven in [24] (see pages 263-274). John Mather discussed a similar result in his Princeton graduate class in the fall of 2000. We sketch modification of their arguments. Numerations of lemmas from there.

1. Represent local invariant manifold of an invariant curve as a graph of a gradient of  $C^{1,1}$ -function. It is usually called an elementary solution. See e.g. Lemma 6.1.
2. Following Fathi [34] represent the barrier function as the difference of elementary solutions. See e.g. Lemma 6.2.
3. Introduce a new parameter  $\sigma \in \mathbb{R}$ , given by a certain oriented area between invariant curves<sup>23</sup>. The family of local invariant manifolds has 1/2-Holder dependence on  $\sigma$  (see formula (6.4), [24]).
4. Show that with respect to two parameters  $c$  and  $\sigma$  the barrier is 1/2-Holder (see Lemma 6.4).

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<sup>23</sup>This step will be discussed in more details later.

5. Prove that the set of barrier functions parametrized by  $c$ 's has box-counting dimension at most 4 (page 273) <sup>24</sup>.
6. Show that parallel translation of the whole family of barrier functions can achieve required nondegeneracy (see Lemma 7.2 [24]) <sup>25</sup>.

Now we discuss steps 3 and 6 in more details.

Step 3. By Key Theorem 1 we have that our NHIC  $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$  (resp.  $\mathcal{M}_{h,\varepsilon}^{E_{j-1}, E_j}$ ) is a Lipschitz graph over  $(\theta^f, p^f, t)$  in the normal form.

Discretize by taking time one map and denote by  $\pi^f$  projection onto  $(\theta^f, p^f)$ . Let  $\gamma_0$  be an invariant curve. We parametrize other invariant curves by  $\sigma$  as follows:

$$\sigma = \int_0^1 (p^f(\gamma_\sigma(\theta^f)) - p^f(\gamma_0(\theta^f))) d\theta^f.$$

Each invariant curve in the domain of definition is uniquely determined by  $\sigma$ . Simple geometric consideration shows that

$$\|\gamma_{\sigma_1} - \gamma_{\sigma_2}\| \leq C|\sigma_1 - \sigma_2|^{1/2},$$

where  $C$  depends on various Lipschitz constants of the cylinder and invariant curves in the Mather graph theorem (see end of page 266 [24]).

This leads through Lemma 6.3 to 1/2-Holder dependence of elementary solutions of invariant curves on  $\sigma$ .

Step 6. We state the corresponding statement about parallel translation of the whole family of barrier functions.

Recall that  $\tilde{h}_c$  is the barrier function defined on the covering space  $(2\mathbb{T})^2 \times \mathbb{R}^2$ ,  $\xi : (2\mathbb{T})^2 \times \mathbb{R}^2 \rightarrow \mathbb{T}^2 \times \mathbb{R}^2$  is the covering map, and  $\tilde{H}$  is the Hamiltonian lifted to the covering space.

Define the generating function  $G(x, x') : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$G(x, x') = \inf_{\gamma(0)=x, \gamma(1)=x'} \int_0^1 L_H(t, \gamma, \dot{\gamma}),$$

where  $L_H$  is the Lagrangian corresponding to  $H$ . A convenient way of to perturb the functions  $\tilde{h}_c$  is by locally perturbing the generating functions. Denote by  $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$  the standard projection.

We consider the following perturbation

$$G'(\theta, \theta') = G(\theta, \theta') + G_1(\theta')$$

---

<sup>24</sup>Estimating Hausdorff dimension of a two parameter family with 1/2-Holder dependence on parameters is not difficult. See remark at the bottom of page 273 [24].

<sup>25</sup>This step will be discussed in more details later.

and denote by  $\tilde{h}'_c$  the corresponding perturbed barrier function. We have the following statement.

**Lemma 11.6.** (*[24], Lemma 7.1 in our notations*) For  $c = \bar{c}_h(E)$  with  $E \in [\bar{E}_j, \bar{E}_{j+1}]$ , the following hold there are two positive radii  $\rho_1, \rho_2 > 0$  with the following properties.

1. There exists a family of open sets  $O(c) \subset \mathbb{T} \times (2\mathbb{T})$  such that the full orbit of any  $(\tilde{\theta}, \tilde{p}, t) \in \tilde{\mathcal{N}}_{\tilde{H}}(c) \setminus \tilde{\mathcal{A}}_{\tilde{H}}(c)$  must intersect  $O(c)$  in the  $\tilde{\theta}$  component.
2. There exists  $\rho_1 > 0$  and  $u \in O(c)$  such that if we perturb  $G(\theta, \theta')$  by a bump function  $G_1(\theta')$  with  $\text{supp } G_1 \subset B_{\rho_1}(u)$ , where  $B_{\rho_1}(u)$  is the ball of radius  $\rho_1$  centered at  $u$ , then for each  $c = \bar{c}_h(E)$  with  $E \in [\bar{E}_j, \bar{E}_{j+1}]$  the corresponding barrier function

$$\tilde{h}'_c(\theta_1, \theta) + \tilde{h}'_c(\theta, \theta_2) = \tilde{h}_c(\theta_1, \theta) + \tilde{h}_c(\theta, \theta_2) + G_1(\theta)$$

for each  $\theta \in O(c)$  and each pair  $\theta_1, \theta_2 \in \xi^{-1}\mathcal{A}_{\tilde{H}}(c)$  such that  $\theta_1$  and  $\theta_2$  belong to different components of  $\mathcal{A}_{\tilde{H}}(c)$ .

3.  $\xi O(c) \cap \{\theta : \|\theta^s - \theta_j^s(c)\| \leq \rho_2\} = \emptyset$ , in particular,  $\xi O(c) \cap \mathcal{N}_{\tilde{H}}(c) = \emptyset$ . Moreover,  $\tilde{U} = \bigcup_{E \in [\bar{E}_j, \bar{E}_{j+1}]} O(c)$  is an open set.

Due to Step 5 the family of barrier functions (37–38) parametrized by cohomology  $c \in \{\bar{c}_h(E)\}_{E \in [\bar{E}_j, \bar{E}_{j+1}]}$  has Hausdorff dimension 4. Using Lemma 11.6 we can translate this family so that it has only isolated minima (see Lemma 7.2). Denote by  $C_0^r$  the set of  $C^r$  functions with a compact support.

**Lemma 11.7.** (*[24], Lemma 7.2 in our notations*) There is a residual<sup>26</sup> set of functions  $G_1 \in C_0^r$  such that with notations of Proposition 11.5 for each  $E \in [\bar{E}_j, \bar{E}_{j+1}]$  and  $c = \bar{c}_h(E)$  we have that the set of minima outside of neighborhood of the lifts of  $\mathcal{A}_{\tilde{H}}(c)$  and of  $\mathcal{A}_{\tilde{H}}(c)$  resp.

$$\tilde{h}_{\bar{c}_h(E)}(\tilde{\theta}_1, \theta) + \tilde{h}_{\bar{c}_h(E)}(\theta, \tilde{\theta}_2) + G_1(\theta)$$

and

$$\tilde{h}_{\bar{c}_h(E)}(\theta', \theta) + \tilde{h}_{\bar{c}_h(E)}(\theta, \theta'') + G_1(\theta)$$

is totally disconnected.

---

<sup>26</sup>Recall that a set of a topological space is residual (or Baire generic) if it contains a countable intersection of open dense subsets.

The nontrivial part of this statement is that the nondegeneracy of these barrier functions can be achieved for all  $E \in [\bar{E}_j, \bar{E}_{j+1}]$  simultaneously. This is an uncountable set. Since the aforementioned non-degeneracy property of the barrier functions is open, it suffices to prove the above lemma only on a small interval.

Notice that once Proposition 11.5 is proven we need to prove that the corresponding cohomology classes  $\Gamma_i^{sr}$  and  $\Gamma_i^{dr}$  for Key Theorems 8 and 9 resp. There are two regimes:

- diffusing along a cylinder without invariant curves.
- diffusing transversally to a cylinder along homolitic orbits or from one cylinder to another along heteroclinic orbits.

In the former case Theorem 0.11 [9] applies directly. In the latter case we apply Theorem 0.12 [9] with the following modification.

Consider the corresponding (discrete) Mañé and Aubry sets  $\mathcal{N}(c)$  and  $\mathcal{A}(c)$ . We proved that  $\mathcal{N}(c) \setminus \mathcal{A}(c)$  is not empty and totally disconnected outside of a neighborhood of  $\mathcal{A}(c)$ .

This is a weaker condition, then  $\mathcal{N}(c) \setminus \mathcal{A}(c)$  being non empty and containing a finitely many orbits. The proof Theorem 0.12 [9] is located in section 9, where the author generalizes this Theorem to the following setting.

Suppose the set  $\mathcal{N}(c) \setminus \mathcal{A}(c)$  is neat, i.e. admits a compact set  $\tilde{\mathcal{K}}(c)$  which contains one and only one point in each orbit of the underlying Hamiltonian system and whose projection  $\pi\tilde{\mathcal{K}}(c)$  onto  $M$  (in our case  $\mathbb{T}^2$ ) is acyclic. This means that  $\pi\tilde{\mathcal{K}}(c)$  has a neighborhood  $U$  whose inclusion into  $M$  induces the null map  $i_* : H_1(U, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ .

The role of a compact set is played by a compact set in the preimage  $\pi^{-1}(O(c))$  of  $O(c)$ . The latter condition holds if  $\mathcal{N}(c) \setminus \mathcal{A}(c)$  is totally disconnected.

## 12 Equivalent forcing class between kissing cylinders

In this section we prove Key Theorem 10. Recall that we have two homology classes  $h, h_1 \in H_1(\mathbb{T}^2, \mathbb{Z})$  such that  $h$  is non-simple, while  $h_1$  is simple critical and properly chosen, i.e.  $\lim_{E \rightarrow 0} \gamma_h^E = \gamma_E^0 = n_1 \gamma_{h_1}^0 + n_2 \gamma_{h_2}^0$  for some  $n_1, n_2 \in \mathbb{Z}_+$  and a simple critical  $\gamma_{h_2}^0$  (see Lemma 3.2 for more details). We would like to prove equivalence of cohomology classes  $c_h(E)$  and  $c_{h_1}(E_1)$  corresponding to there cohomologies. The proof consists of four steps. In section B around a (strong) double resonance we define a slow mechanical system  $H^s(\varphi^s, I^s) = K(I^s) - U(\varphi^s)$ .

1. We construct a special variational problem for the slow mechanical system  $H^s$ . A solution of this variational problem is an orbit “jumping” from one homology class  $h$  to the other  $h_1$ . The same can be done with  $h$  and  $h_1$  switched.
2. We modify this variational problem for the fast time-periodic perturbation of  $H^s$ , i.e. for the perturbed slow system  $H_\varepsilon^s(\varphi^s, I^s, \tau) = K(I^s) - U(\varphi^s) + \sqrt{\varepsilon} P(\varphi^s, I^s, \tau)$  with  $\tau \in \sqrt{\varepsilon} \mathbb{T}$ .  
Recall the original Hamiltonian system  $H_\varepsilon$  near a double resonance can be brought to a normal form  $N_\varepsilon = H_\varepsilon \circ \Phi_\varepsilon$  and this normal form, in turn, is related to the perturbed slow system through an affine coordinate change and two rescalings (see section B).
3. We adapt and modify the latter variational problem and prove that its solution is an orbit connecting different homologies  $h$  and  $h_1$ .
4. Using this variational problem we prove forcing relation between  $c_h(E)$  and  $c_{h_1}^{e_0}(E)$ .

### 12.1 Variational problem for the slow mechanical system

Recall that the slow mechanical system is given by

$$H^s(\varphi^s, I^s) = K(I^s) - U(\varphi^s),$$

and let  $m_0$  denote the point achieving the minimum of  $U$ . Given  $m \in \mathbb{T}^2$ ,  $a > 0$  and a unit vector  $\omega \in \mathbb{R}^2$ , define

$$S(m, a, \omega) = \{m + \lambda \omega : \lambda \in (-a, a)\}.$$

$S(m, a, \omega)$  is a line segment in  $\mathbb{T}^2$  and we will refer to it as a *section* (see Figure 22).

In Proposition 4.2 we defined the cohomology class  $\bar{b}_{h_1}(E)$  satisfying the condition

$$\lim_{E \rightarrow 0+} \frac{\bar{b}_{h_1}(E) - \bar{c}_h(E)}{\|\bar{b}_{h_1}(E) - \bar{c}_h(E)\|} = h_1^\perp.$$

The cohomology class  $\bar{b}_{h_1}^e(E)$  is a modification of  $\bar{b}_{h_1}(E)$  for  $0 \leq E \leq e$  (see also Figure 14).

**Proposition 12.1.** *Suppose the slow mechanical system  $H^s$  satisfies conditions [A0]-[A4]. Then there exists  $e_0 > 0$  such that the following hold. For each  $0 \leq E \leq e_0$ , there exists a section  $S(E) := S(m(E), a(E), \omega(E))$ , satisfying the following conditions:*

1. *For some  $a > 0$  we have  $a(E) \geq a$ .*
2.  *$m(E)$  can be chosen so that  $m(E) \rightarrow m_0$ .*
3. *We also have*

$$\alpha_{H^s}(\bar{c}_h(E)) = \alpha_{H^s}(\bar{b}_{h_1}(E)), \quad (\bar{c}_h(E) - \bar{b}_{h_1}(E)) \cdot \omega(E) = 0.$$

4. *There exists a compact set  $K(E) \subset S(E)$  such that for all  $x^E \in \mathcal{A}_{H^s}(\bar{c}_h(E))$  and  $z^E \in \mathcal{A}_{H^s}(\bar{b}_{h_1}(E))$ , the minimum of the variational problem*

$$\min_{y \in \overline{S(E)}} \left\{ h_{\bar{c}_h(E)}(x^E, y) + h_{\bar{b}_{h_1}(E)}(y, z^E) \right\} \quad (39)$$

*is never reached outside of  $K(E)$ .*

5. *Assume that the minimum in (39) is reached at  $y_0$ . Let  $p_1 - \bar{c}_h(E)$  be any super-differential of  $h_{\bar{c}_h(E)}(x^E, \cdot)$  at  $y_0$  and  $-p_2 + \bar{b}_{h_1}(E)$  be any super-differential of  $h_{\bar{b}_{h_1}(E)}(\cdot, z^E)$  at  $y_0$ , then*

$$\partial_{\varphi^s} H^s(y_0, p_1) \cdot (\bar{c}_h(E) - \bar{b}_{h_1}(E)) \quad \text{and} \quad \partial_{\varphi^s} H^s(y_0, p_2) \cdot (\bar{c}_h(E) - \bar{b}_{h_1}(E))$$

*have the same signs.*

Moreover, the same conditions are satisfied with  $\bar{c}_h(E)$  and  $\bar{b}_{h_1}(E)$  switched.

**Remark 12.1.** 1. While condition 5 in Proposition 12.1 is stated in terms of super-differentials, it can be understood as a statement on the velocity of the minimizers. More precisely, let  $\gamma_1 : (-\infty, 0) \rightarrow \mathbb{T}^2$  be a backward minimizer for  $h_{\bar{c}_h(E)}(x^E, y)$ , and  $\gamma_2 : (0, \infty) \rightarrow \mathbb{T}^2$  a forward minimizer for  $h_{\bar{b}_{h_1}(E)}(y, z^E)$  (see section 10.1), then velocities satisfy

$$\dot{\gamma}_1(0) = \partial_{\varphi^s} H^s(y_0, p_1), \quad \dot{\gamma}_2(0) = \partial_{\varphi^s} H^s(y_0, p_2).$$

In this sense, condition 5 implies that the minimizers cross the section  $S(E)$  in the same direction instead of “turning back”.



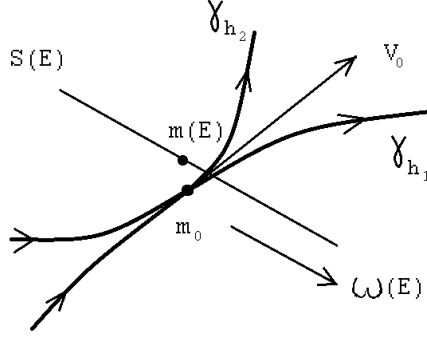


Figure 22: Jump from one cylinder to another in the same homology

2. Conditions 3–5 imply the so-called “no corners” condition. In the language of the minimizers, conditions 3–5 imply  $\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$ , and as a consequence, the minimizers  $\gamma_1$  and  $\gamma_2$  concatenate to a smooth orbit of the Euler-Lagrange flow.

*Proof of Proposition 12.1.* We will first prove the statements for  $E = 0$  and use continuity for  $E > 0$ .

Recall that  $\mathcal{A}_{H^s}(\bar{c}_h(0)) = \gamma_{h_1}^0 \cup \gamma_{h_2}^0$ , and the curves  $\gamma_{h_1}^0$  and  $\gamma_{h_2}^0$  are tangent to a common direction at  $m_0$ , which we will call  $v_0$ . By the choice of  $h_1$ ,  $v_0$  is not orthogonal to  $h_1^\perp$ . We now explain the choice of the section  $S(m(E), a(E), \omega(E))$ . Define  $\omega(E)$  to be a unit vector orthogonal to  $\bar{c}_h(E) - \bar{b}_{h_1}(E)$ . For a sufficiently small  $e_0$ , we have that  $\omega(E)$  is transversal to  $v_0$  for all  $0 \leq E \leq e_0$ . As a consequence, for any  $m(E)$  sufficiently close to  $m_0$ , there exists  $a(E) > 0$  such that the section  $S(m(E), a(E), \omega(E))$  intersects  $\gamma_{h_1}^0 \cup \gamma_{h_2}^0$  transversally. All functions  $m(E)$ ,  $a(E)$  and  $\omega(E)$  can be chosen to be continuous, with  $m(0) = 0$ ,  $a(0) = a > 0$  and  $\omega(0) = v$  (see Figure 22). These definitions imply conditions 1–3 of Proposition 12.1.

Note that  $\bar{c}_h(0) = \bar{b}_{h_1}(0)$ . The Aubry set  $\tilde{\mathcal{A}}_{H^s}(\bar{c}_h(0)) = \tilde{\mathcal{A}}_{H^s}(\bar{b}_{h_1}(0))$  supports a unique invariant measure, which is the saddle fixed point. As a consequence, the Aubry set has a unique static class. Hence for any  $x^0 \in \mathcal{A}_{H^s}(\bar{c}_h(0))$  and  $z^0 \in \mathcal{A}_{H^s}(\bar{b}_{h_1}(0))$ , the minimum in

$$\min_{y \in S(0)} \left\{ h_{\bar{c}_h(0)}(x^0, y) + h_{\bar{b}_{h_1}(0)}(y, z^0) \right\}$$

is achieved at  $S(0) \cap (\gamma_{h_1}^0 \cup \gamma_{h_2}^0)$ . This implies condition 4.

Moreover, by Proposition 10.4, part 3, we know that the barrier functions  $h_{\bar{c}_h(0)}(x^0, \cdot)$  and  $h_{\bar{b}_{h_1}(0)}(\cdot, z^0)$  are both differentiable at  $y_0 \in S(0) \cap (\gamma_{h_1}^0 \cup \gamma_{h_2}^0)$ . Assume that

$p_1 - \bar{c}_h(0)$  and  $-p_2 + \bar{b}_{h_1}(0)$  be the derivatives, then

$$\partial_{\varphi^s} H^s(y_0, p_1) \quad \text{and} \quad \partial_{\varphi^s} H^s(y_0, p_2)$$

both equals the velocity of  $\gamma_{h_1}$  or  $\gamma_{h_2}$  as they cross the section. This implies condition 5.

Since conditions 4–5 are satisfied for  $E = 0$ , by Proposition 10.7, they are also satisfied for a sufficiently small  $E > 0$ .  $\square$

## 12.2 Variational problem in the original coordinates

The original Hamiltonian  $H_\epsilon$  is conjugate to the perturbed slow system

$$H_\epsilon^s(\varphi^s, I^s, \tau) = H_0(p_0)/\epsilon + K(I^s) - U(\varphi^s) + \sqrt{\epsilon} P(\varphi^s, I^s, \tau).$$

We will first describe a variational problem for the system  $H_\epsilon^s$ , then translate it into a variational problem for the system  $H_\epsilon$ . It is no longer true that  $\alpha_{H_\epsilon^s}(\bar{c}_h(E)) = \alpha_{H_\epsilon^s}(\bar{b}_{h_1}(E)) = E$ , instead, we have the following lemma:

**Lemma 12.2.** *Fix  $e_0 > 0$ . There exists  $C > 0$ , and  $\epsilon_0 > 0$  such that for any  $\frac{e_0}{3} \leq E \leq \frac{2e_0}{3}$  and  $0 \leq \epsilon \leq \epsilon_0$ , there exists  $0 < E^\epsilon < e_0$  such that*

$$\alpha_{H_\epsilon^s}(\bar{c}_h(E)) = \alpha_{H_\epsilon^s}(\bar{b}_{h_1}(E^\epsilon)), \quad |E - E^\epsilon| \leq C\sqrt{\epsilon}.$$

Choose  $e = e_0$ , we define  $\bar{c}_{h_1}^\epsilon(E)$  as in section 4.3. By definition,  $\bar{c}_{h_1}^\epsilon(E) = \bar{b}_{h_1}(E)$ . We choose  $\omega^\epsilon(E)$  to be a unit vector orthogonal to  $\bar{c}_h(E) - \bar{c}_{h_1}^\epsilon(E)$ , and the section  $S^\epsilon(E) = S(m(E), a(E), \omega^\epsilon(E))$ . We have:

- $\alpha_{H_\epsilon^s}(\bar{c}_h(E)) = \alpha_{H_\epsilon^s}(\bar{c}_{h_1}^\epsilon(E^\epsilon))$  and  $(\bar{c}_h(E) - \bar{c}_{h_1}^\epsilon(E^\epsilon)) \cdot \omega^\epsilon(E) = 0$ .
- As  $\epsilon$  approaches 0,  $E^\epsilon \rightarrow E$ ,  $\bar{c}_{h_1}^\epsilon(E^\epsilon) \rightarrow \bar{c}_{h_1}^\epsilon(E)$ , and  $\omega^\epsilon(E) \rightarrow \omega(E)$ .

The analog of (39) for  $H_\epsilon^s$  is given by

$$\min_{y \in S^\epsilon(E), \tau \in \sqrt{\epsilon}\mathbb{T}} \left\{ h_{H_\epsilon^s, \bar{c}_h(E)}(x^{E,\epsilon}, 0; y, \tau) + h_{H_\epsilon^s, \bar{c}_{h_1}^\epsilon(E)}(y, \tau; z^{E,\epsilon}, 0) \right\}, \quad (40)$$

where  $(x^{E,\epsilon}, 0) \in \mathcal{A}_{H_\epsilon^s}(\bar{c}_h(E))$  and  $(z^{E,\epsilon}, 0) \in \mathcal{A}_{H_\epsilon^s}(\bar{c}_{h_1}^\epsilon(E))$ . Note that the Aubry sets  $\tilde{\mathcal{A}}_{H^s}(\bar{c}_h(E))$  and  $\tilde{\mathcal{A}}_{H^s}(\bar{c}_{h_1}^\epsilon(E))$  are both supported on a single periodic orbit, and hence supports a unique minimal measure. Moreover, by Proposition B.5,  $H_\epsilon^s$  Tonelli converges to  $H^s$ . By Corollary 10.7, the time periodic variational problem (40) converges to the slow variational problem (39) as  $\epsilon \rightarrow 0$ .

We now convert the variational problem (40) to the original system, and study its relation with (39).

Recall that the original Hamiltonian  $H_\epsilon$  can be brought into a normal form system  $N_\epsilon$ . The perturbed slow system  $H_\epsilon^s$  and the normal form system  $N_\epsilon$  are related through an affine coordinate change  $\Phi_L$  and two rescaling operators  $\mathcal{S}_2 \circ \mathcal{S}_1$ , see section B.

Denote

$$\Phi_L^1(\varphi^s, t) = \begin{bmatrix} B^{-1} & -B^{-1} \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi^s \\ t \end{bmatrix}, \quad (\Phi_L^1)^{-1}(\theta, t) = \begin{bmatrix} B & \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ t \end{bmatrix},$$

this is the angular component of the affine coordinate change  $\Phi_L$ . The  $2 \times 2$  matrix  $B$  is defined in (51). According to Propositions B.6 and B.7, the following relations hold for the Aubry sets and the barrier functions of  $H_\epsilon^s$  and  $N_\epsilon$ .

Recall that the functions  $c_h(E)$  and  $\bar{c}_h(E)$  are defined by the relation

$$c_h(E) = p_0^s + \bar{c}_h(E)\sqrt{\epsilon}.$$

Let  $\theta_0$  be the unique minimum of the function  $-Z(\theta, p_0)$ , we have that  $\theta_0 = B^{-1}(m_0)$ . A section  $S(m, a, \omega) \times \sqrt{\epsilon}\mathbb{T} \subset \mathbb{T}^2 \times \sqrt{\epsilon}\mathbb{T}$  is mapped under  $\Phi_L^1$  to

$$\Sigma(\theta_1, a, \Omega, l) = \{(\theta_1 + \lambda\Omega + lt, t) \in \mathbb{T}^2 \times \mathbb{T} : -a < \lambda < a, t \in \mathbb{T}\},$$

where  $\theta_1 = B^{-1}(m)$ ,  $\Omega = B^{-1}\omega$ , and  $l = -B^{-1}(k_0, k'_0) \in \mathbb{Z}^2$ . We define  $\Sigma^\epsilon(E)$  using the section  $S^\epsilon(E) \times \sqrt{\epsilon}\mathbb{T}$ . We now state the variational problem for the original coordinates.

**Proposition 12.3.** *Let  $c_h$  and  $c_{h_1}^\epsilon$  and  $E$  be defined as before. There exist  $\epsilon_0 > 0$  and  $b > 0$  such that for  $0 \leq \epsilon \leq \epsilon_0$ , there exist  $0 < E^\epsilon \leq e_0$  and a section  $\Sigma^\epsilon(E) := \Sigma(\theta_1(E), a(E), \Omega^\epsilon(E), l)$ , satisfying the following conditions:*

1. *For some  $a > 0$  we have  $a(E) \geq a$ .*
2.  *$m(E)$  can be chosen so that  $m(E) \rightarrow m_0$ .*
3. *We also have*

$$\begin{aligned} \alpha_{H_\epsilon}(c_h(E)) - \alpha_{H_\epsilon}(c_{h_1}^\epsilon(E^\epsilon)) &= -(c_h(E) - c_{h_1}^\epsilon(E^\epsilon)) \cdot l, \\ (c_h(E) - c_{h_1}^\epsilon(E^\epsilon)) \cdot \Omega^\epsilon(E) &= 0. \end{aligned}$$

4. *There exists a compact set  $K$  such that for all  $(\chi^{E,\epsilon}, 0) \in \mathcal{A}_{H_\epsilon}(c_h(E))$  and  $(\xi^{E,\epsilon}, 0) \in \mathcal{A}_{H_\epsilon}(c_{h_1}^\epsilon(E^\epsilon))$ , the minimum of the variational problem*

$$\min_{(\psi, t) \in \Sigma^\epsilon(E)} \left\{ h_{H_\epsilon, c_h(E)}(\chi^{E,\epsilon}, 0; \psi, t) + h_{H_\epsilon, c_{h_1}^\epsilon(E)}(\psi, t; \xi^{E,\epsilon}, 0) \right\} \quad (41)$$

*is never achieved outside of  $K$ .*

5. Assume that the minimum in condition 2 is reached at  $(\psi_0, t_0)$ . Let  $p_1^\epsilon - c_h(E)$  and  $-p_2^\epsilon + c_{h_1}^\epsilon(E^\epsilon)$  be super-differentials of the barrier functions  $h_{H_\epsilon, c_h(E)}(\chi^{E, \epsilon}, 0; \cdot, t_0)$  and  $h_{H_\epsilon, c_{h_1}(E)}(\cdot, t_0; \xi^{E, \epsilon}, 0)$  respectively. Then

$$(\partial_p H_\epsilon(\psi_0, p_1^\epsilon, t_0) - l) \cdot (c_h(E) - c_{h_1}^\epsilon(E^\epsilon)), \quad (\partial_p H_\epsilon(\psi_0, p_2^\epsilon, t_0) - l) \cdot (c_h(E) - c_{h_1}^\epsilon(E^\epsilon))$$

have the same signs.

Moreover, the same conditions are satisfied with  $c_h(E)$  and  $c_{h_1}^\epsilon(E^\epsilon)$  switched.

**Theorem 24.** Assume that the conclusions of Proposition 12.3 hold. In addition, assume that both  $\mathcal{A}_{H_\epsilon}(c_h(E))$  and  $\mathcal{A}_{H_\epsilon}(c_{h_1}^\epsilon(E^\epsilon))$  admits a unique static class. Then

$$c_h(E) \dashv\vdash c_{h_1}^\epsilon(E^\epsilon).$$

We prove Key Theorem 10 assuming Proposition 12.3, Theorem 10 and Theorem 23 in section 11.

*Proof of Key Theorem 10.* By Proposition 12.3, for the system  $\bar{H}_\epsilon$ , which is a perturbation of  $H_\epsilon$ , all conditions of Theorem 24 are satisfied, except the condition of uniqueness of static classes.

The uniqueness of static class is satisfied by the particular choice of perturbations. It is proved in Theorem 23 that for the perturbed system, for each  $c_h(E)$  and  $c_{h_1}^\epsilon(E^\epsilon)$ , the associated  $c$ -minimal measure is unique. This implies uniqueness of static class.  $\square$

We prove Proposition 12.3 in section 12.3 and prove Theorem 10 in section 12.4.

## 12.3 Scaling limit of the barrier function

In this section we prove Proposition 12.3. It is readily verified that conditions 1–3 are satisfied by our choice of the section  $\Sigma^\epsilon$ . It remains to prove conditions 4 and 5.

We will show that the variational problem (39) is a scaling limit of the variational problem (41).

**Proposition 12.4.** The family of functions  $h_{H_\epsilon, c_h(E)}(\chi^{E, \epsilon}, 0; \cdot, t)/\sqrt{\epsilon}$  is uniformly semi-concave, and

$$\lim_{\epsilon \rightarrow 0+} \inf_{C \in \mathbb{R}} \sup_{(y, t) \in \mathbb{T}^2 \times \mathbb{T}} \left| h_{H_\epsilon, c_h(E)}(\chi^{E, \epsilon}, 0; \psi, t)/\sqrt{\epsilon} - h_{H^s, \bar{c}_h(E)}(x^E, B\psi + (k_0, k'_0)t) - C \right| = 0.$$

uniformly over  $(\chi^{E, \epsilon}, 0) \in \mathcal{A}_{H_\epsilon}(c_h(E))$  and  $(x^E, 0) \in \mathcal{A}_{H^s}(\bar{c}_h(E))$ .

Moreover, the super-differential  $\partial_y h_{H_\epsilon, c_h(E)}(\chi^{E,\epsilon}, 0; \psi, t)/\sqrt{\epsilon}$  converges uniformly to a super-differential of the limit, in the sense that

$$\lim_{\epsilon \rightarrow 0} \inf_{l_\epsilon \in \partial_\psi h_{H_\epsilon, c_h(E)}(\chi^{E,\epsilon}, 0; \psi, t)/\sqrt{\epsilon}} d(l_n, \partial_\psi h_{H^s, c_h(E)}(x^E, B\psi + (k_0, k'_0)t)) = 0$$

uniformly over  $\chi^{E,\epsilon}$  and  $x^E$ . The same conclusions apply to the barrier function  $h_{H_\epsilon, c_{h_1}^\epsilon(E)}(\cdot, t; \xi^{E,\epsilon}, 0)/\sqrt{\epsilon}$ .

To prove Proposition 12.4, we first state some relations between the Aubry sets and the barrier functions for the system  $H_\epsilon$  and the perturbed slow system  $H_\epsilon^s$ .

**Proposition 12.5.** *Assume that the cohomologies  $c$  and  $\bar{c}$  satisfy*

$$\bar{c} = (B^T)^{-1}(c - p_0)/\sqrt{\epsilon}. \quad (42)$$

Then the following hold:

- $\alpha_{H_\epsilon}(c) = \epsilon \alpha_{H_\epsilon^s}(\bar{c}) - \sqrt{\epsilon} \bar{c} \cdot (k_0, k'_0)$ .
- $d(\mathcal{A}_{H_\epsilon}(c), B^{-1}\mathcal{A}_{H_\epsilon^s}(\bar{c})) = O(\epsilon)$ , where  $d$  is Hausdorff distance between sets in  $\mathbb{T}^2 \times \mathbb{T}$ .
- For  $(\theta_i, t_i) = \Phi_L^1(\varphi_i^s, \tau_i/\sqrt{\epsilon})$ ,  $i = 1, 2$ , we have

$$h_{H_{\epsilon,c}}(\theta_1, t_1; \theta_2, t_2) = h_{H_{\epsilon,\bar{c}}^s}(\varphi_1, \tau_1; \varphi_2, \tau_2) \cdot \sqrt{\epsilon} + O(\epsilon).$$

Proposition 12.5 is a consequence of Propositions B.6, B.7 and B.8. In addition, the following is known about semi-concavity of the barrier function, for a nearly integrable system.

**Proposition 12.6** ([13], Proposition 5.4). *Let  $B \subset H^1(\mathbb{T}^2, \mathbb{R})$  be a bounded set, there is a constant  $K > 0$  and  $\epsilon_0 > 0$ , such that all weak KAM solutions of  $L_{H_\epsilon, c}$  for  $c \in B$  and  $0 \leq \epsilon < \epsilon_0$  are  $K\sqrt{\epsilon}$ -semi-concave.*

*Proof of Proposition 12.4.* By Proposition 12.5 and Proposition 12.6, we have that the family of functions  $h_{H_\epsilon, c_h(E)}(\chi^{E,\epsilon}, 0; \psi, t)/\sqrt{\epsilon}$  is uniformly semi-concave, and converges to the same limit as

$$h_{H_\epsilon^s}(x^{E,\epsilon}, 0; y, \tau)$$

as  $\epsilon \rightarrow 0$ . Here  $x^{E,\epsilon} \in \mathcal{A}_{H_\epsilon^s}$  and  $(y, \tau/\sqrt{\epsilon}) = (\Phi_L^1)^{-1}(\psi, t)$ . By Corollary 10.7, the functions  $h_{H_\epsilon^s}(x^{E,\epsilon}, 0; y, \tau)$  converges uniformly to  $h_{H^s}(x^E, y)$  as  $\epsilon \rightarrow 0$ . This proves uniform convergence.

The convergence of super-differential is proved in the same way as the second part of Corollary 10.7.  $\square$

Proposition 12.3 immediately follows from Proposition 12.4 and Proposition 12.1.

## 12.4 Proof of forcing relation

In this section we prove Theorem 24. We fix a Tonelli Hamiltonian  $H : T^*\mathbb{T}^2 \times \mathbb{T} \rightarrow \mathbb{R}$  and drop all subscript indicating the Hamiltonian. We rephrase the three conditions in Theorem 12.3 as follows:

There exists cohomologies  $c_1, c_2$  and a section  $\Sigma(\theta_1, a, \Omega, l)$  such that the following hold.

N1.  $\alpha(c_1) - \alpha(c_2) = -(c_1 - c_2) \cdot l, (c_1 - c_2) \cdot \Omega = 0.$

N2. There exists a compact set  $K$  such that for any  $x \in \mathcal{A}(c_1)$  and  $z \in \mathcal{A}(c_2)$ , the minimum of the variational problem

$$\min_{(y,t) \in \Sigma} \{h_{c_1}(x, 0; y, t) + h_{c_2}(y, t; z, 0)\}$$

is never taken outside of  $K$ .

N3. Assume that the minimum N2 is reached at  $(y_0, t_0)$ , and let  $p_1 - c_1$  and  $-p_2 + c_2$  be any super-differentials of  $h_{c_1}(x, 0; \cdot, t)$  and  $h_{c_2}(\cdot, t_0; z, 0)$  respectively. Then

$$(\partial_p H(y_0, p_1, t_0) - l) \cdot (c_1 - c_2), \quad (\partial_p H(y_0, p_2, t_0) - l) \cdot (c_1 - c_2)$$

have the same signs.

N4. Both  $\mathcal{A}(c_1)$  and  $\mathcal{A}(c_2)$  contains a unique static class.

The following statement implies Theorem 24.

**Proposition 12.7.** *Assume that  $c_1, c_2$  and  $\Sigma$  satisfies the conditions N1–N4. Then the following hold.*

1. (interior minimum) *There exists  $N < N', M < M' \in \mathbb{N}$  and a compact set  $K' \subset \Sigma$ , such that for any semi-concave function  $u$  on  $\mathbb{T}^2$ , the minimum in*

$$v(z) := \min\{u(x) + A_{c_1}(x, 0; y, t + n) + A_{c_2}(y, t + n; z, n + m)\}, \quad (43)$$

*where the minimum is taken in*

$$x \in \mathbb{T}^2, (y, t) \in \Sigma, N \leq n \leq N', M \leq m \leq M',$$

*is never achieved for  $(y, t) \notin K'$ .*

2. (no corner) *Assume the minimum in (43) is achieved at  $(y, t) = (y_0, t_0)$ ,  $(n, m) = (n_0, m_0)$ , and the minimizing curves are  $\gamma_1 : [0, t_0 + n_0] \rightarrow \mathbb{T}^2$  and  $\gamma_2 : [t_0 + n_0, t_0 + n_0 + m_0] \rightarrow \mathbb{T}^2$ . Then  $\gamma_1$  and  $\gamma_2$  connect to an orbit of the Euler-Lagrange equation, i.e.*

$$\dot{\gamma}_1(t_0 + n_0) = \dot{\gamma}_2(t_0 + n_0).$$

3. (connecting orbits) The function  $v$  is semi-concave, and its associated pseudograph satisfies

$$\overline{\mathcal{G}_{c_2, v}} \subset \bigcup_{0 \leq t \leq N' + M'} \varphi^t \mathcal{G}_{c_1, u}.$$

As a consequence,

$$c_1 \vdash c_2.$$

**Remark 12.2.** This Proposition represents a more sophisticated version of Proposition 5.1. Points  $x \in \mathbb{T}^2$ , where minimum is achieved for some  $(y, t) \in K'$ ,  $z \in \mathbb{T}^2$ ,  $n, m$ , are points of differentiability of  $u$ . At such points the pseudograph  $\mathcal{G}_{c_1, u_1}$  is well-defined. Similarly, to Proposition 5.1 we prove that for each minima  $x_0 \in \mathbb{T}^2$ ,  $(y_0, t_0) \in K'$ ,  $N \leq n_0 \leq N'$ ,  $M \leq m_0 \leq M'$  we have  $\varphi^{n_0 + m_0}(x_0, du + c_1) = (z, dv + c_2)$ .

Taking  $N$  and  $M$  large forces solutions to this variational problem to start at some  $x_0$ , then approaches  $\mathcal{A}(c_1)$  and spend long time nearby, then approach  $\mathcal{A}(c_2)$  and also spend long time nearby. Thus, the corresponding solutions approach to some heteroclinic orbits connecting  $\mathcal{A}(c_1)$  and  $\mathcal{A}(c_2)$ .

Conclusion 1 from Proposition 12.7 is a finite time version of condition N2. In order to prove this statement, we need a uniform convergence property of the function  $A_c$  to the barrier function  $h_c$ , and a characterization of  $h_c$ .

**Lemma 12.8.** 1. Let  $u$  be a continuous function on  $\mathbb{T}^2$ . The limit

$$\begin{aligned} \lim_{N \rightarrow \infty} \lim_{N' \rightarrow \infty} \min_{x \in \mathbb{T}^2, N \leq n \leq N'} \{u(x) + A_c(x, 0; y, t + n)\} = \\ = \min_{x \in D} \{u(x) + h_c(x, 0; y, t)\} \end{aligned}$$

is uniform in  $u$  and  $(y, t)$ .

2. The limit

$$\lim_{N \rightarrow \infty} \lim_{N' \rightarrow \infty} \min_{N \leq n \leq N'} A_c(y, t; z, n) = h_c(y, t; z, 0)$$

is uniform in  $y, t, z$ .

*Proof.* The proof of the first item is similar to the proof of Proposition 6.3 of [9] with some auxiliary facts proven in Appendix A there. The proof of the second item is similar to that of Proposition 6.1 from [9].

In both cases the action function, defined in (2.4) and (6.1) of [9], is restricted to have integer time increment. For non-integer time increments the same argument applies.  $\square$

Using Lemma 10.8 and Lemma 10.9, we have the following characterization of the barrier functions.

**Lemma 12.9.** *Assume that  $\mathcal{A}(c)$  has only one static class. For each point  $(y, t) \in \mathbb{T}^2 \times \mathbb{T}$  and each  $z \in \mathbb{T}^2$*

1. *there exists  $x_0 \in \mathbb{T}^2$  and  $x_1 \in \mathcal{A}(c)$  such that*

$$\min_{x \in \mathbb{T}^2} \{u(x) + h_c(x, 0; y, t)\} = u(x_0) + h_c(x_0, 0; x_1, 0) + h_c(x_1, 0; y, t).$$

2. *there exists  $z_1 \in \mathcal{A}(c)$  such that*

$$h_c(y, t; z, 0) = h_c(y, t; z_1, 0) + h_c(z_1, 0; z, 0).$$

To prove the second conclusion of Proposition 12.7, we need a characterization of the super-differentials of the function  $A_c$  obtained in Propositions 10.1 part 3, and 10.2.

*Proof of Proposition 12.7.* According to Lemma 12.8, (43) converges uniformly as  $N, M \rightarrow \infty$  to

$$\min_{x, y, t} \{u(x) + h_{c_1}(x, 0; y, t) + h_{c_2}(y, t; z, 0)\},$$

which is equal to

$$\begin{aligned} & \min_{(y, t)} \{u(x_0) + h_{c_1}(x_0, 0; x_1, 0) + h_{c_1}(x_1, 0; y, t) + h_{c_2}(y, t; z_1, 0) + h_{c_2}(z_1, 0; z, 0)\} \\ &= \min_{(y, t)} \{const + h_{c_1}(x_1, 0; y, t) + h_{c_2}(y, t; z_1, 0) + h_{c_2}(z_1, 0; z, 0)\}. \end{aligned}$$

by Lemma 12.9. Since the above variational problem has a interior minimum due to condition N2, by uniform convergence, the finite-time variational problem (43) also has an interior minimum for sufficiently large  $N, M$ .

We now prove the second conclusion. Let  $\gamma_1$  and  $\gamma_2$  be the minimizers for  $A_{c_1}(x_0, 0; y_0, t_0 + n_0)$  and  $A_{c_2}(y_0, t_0 + n_0; z, n_0 + m_0)$ , and let  $p_1$  and  $p_2$  be the associated momentum, we will show that

$$p_1(t_0 + n_0) = p_2(t_0 + n_0),$$

which implies  $\dot{\gamma}_1(t_0 + n_0) = \dot{\gamma}_2(t_0 + n_0)$ . To abbreviate notations, we write  $p_1^0 = p_1(t_0 + n_0)$  and  $p_2^0 = p_2(t_0 + n_0)$  for the rest of the proof.

Note that

$$u_1(x_0) + A_{c_1}(x_0, 0; y_0, t_0 + n_0) = \min_{x \in \mathbb{T}^2} \{u_1(x) + A_{c_1}(x, 0; y_0, t_0 + n_0)\}.$$



By semi-concavity, the function  $u_1(x) + A_{c_1}(x, 0; y_0, t_0 + n_0)$  is differentiable at  $x_0$  and the derivative vanishes. By Proposition 10.1 part 3,

$$d_x(u_1)(x_0) = p_1(0) - c_1. \quad (44)$$

By a similar reasoning, we have

$$\begin{aligned} & A_{c_1}(x_0, 0; y_0, t_0 + n_0) + A_{c_2}(y_0, t_0 + n_0; z, n_0 + m_0) \\ &= \min_{(y,t) \in \bar{\Sigma}} \{A_{c_1}(x_0, 0; y, t + n_0) + A_{c_2}(y, t + n_0; z, n_0 + m_0)\}. \end{aligned}$$

By semi-concavity, we have

$$\begin{aligned} 0 &= d_y(A_{c_1}(x_0, 0; y, t_0 + n_0) + A_{c_2}(y, t_0 + n_0; z, n_0 + m_0))|_{y=y_0} \cdot \Omega \\ &= (p_1^0 - c_1 - p_2^0 + c_2) \cdot \Omega = (p_1^0 - p_2^0) \cdot \Omega. \end{aligned}$$

Hence

$$p_1^0 - p_2^0 \in \mathbb{R}(c_1 - c_2).$$

We proceed to prove  $p_1^0 = p_2^0$ .

By the definition of  $\Sigma(\theta_1, a, \Omega, l)$ , the vector  $(l, 1)$  is tangent to  $\Sigma$ . As a consequence,

$$\begin{aligned} 0 &= d_{(y,t)}(A_{c_1}(x_0, 0; y, t + n_0) + A_{c_2}(y, t + n_0; z, n_0 + m_0))|_{t=t_0} \cdot (l, 1) \\ &= (p_1^0 - p_2^0) \cdot l - (c_1 - c_2) \cdot l - H(y_0, p_1^0, t_0) - \alpha(c_1) + H(y_0, p_2^0, t_0) + \alpha(c_2) \\ &= -H(y_0, p_1^0, t_0) + H(y_0, p_2^0, t_0) + (p_1^0 - p_2^0) \cdot l, \end{aligned}$$

where the last equality uses condition N1.

Since the function  $H(y_0, p, t_0) - p \cdot l$  is convex in  $p$ , the equation (in  $\lambda$ )

$$H(y_0, p_1^0, t_0) - p_1^0 \cdot l = H(y_0, p_1^0 + \lambda(c_1 - c_2), t_0) - (p_1^0 + \lambda(c_1 - c_2)) \cdot l,$$

has at most two solutions, one of which is  $\lambda = 0$ . If  $\lambda = 0$ , we have  $p_1^0 = p_2^0$ .

To rule out the other possibility, we observe the following simple property of a one-variable convex function. For  $f$  convex, if the equation  $f(\lambda) = a$  has two solutions  $\lambda_1$  and  $\lambda_2$ , then  $f'(\lambda_1)$  and  $f'(\lambda_2)$  have different signs. Indeed, by Rolle's theorem, there is  $\lambda_0 \in (\lambda_1, \lambda_2)$  with  $f'(\lambda_0) = 0$ , and  $f'$  is a monotone function. Apply this observation to  $H(y_0, p_1^0 + \lambda(c_1 - c_2), t_0) - (p_1^0 + \lambda(c_1 - c_2)) \cdot l$ , we conclude that if  $p_2^0 = p_1^0 + \lambda(c_1 - c_2)$ , with  $\lambda \neq 0$ , then

$$(\partial_p H(y_0, p_1, t_0) - l) \cdot (c_1 - c_2), \quad (\partial_p H(y_0, p_2, t_0) - l) \cdot (c_1 - c_2)$$

have different signs. This is not possible due to condition N3. Since coincidence of momentum for the Hamiltonian flow is equivalent to the coincidence of velocity for the Euler-Lagrange flow, the second conclusion follows.

As a consequence,  $(\gamma_1, p_1)$  and  $(\gamma_2, p_2)$  connect as a solution of the Hamiltonian flow. Using (44), we have

$$\varphi^{n_0+m_0}(x_0, du_x(x_0) + c_1) = (z, p_2(n_0 + m_0)).$$

Note that  $p_2(n_0 + m_0) - c_2$  is a super-differential to  $v$  at  $z$ . If  $v$  is differentiable at  $z$ , then  $p_2 = dv(z) + c_2$ . This implies

$$\overline{\mathcal{G}_{c_2, v}} \subset \bigcup_{0 \leq t \leq N' + M'} \varphi^t \mathcal{G}_{c_1, u}$$

and the forcing relation. □

## A Generic properties of mechanical systems on the two-torus

Most of this section is devoted to proving Theorem 4. At the end of the section we prove Theorem 5.

Fix a homology class  $h \in H_1(\mathbb{T}^2, \mathbb{R})$ . We call a periodic orbit of the Hamiltonian system (globally)  $\rho_E$ -minimal, if it is associated to a shortest geodesic curve for the Jacobi metric  $\rho_E$  in the homology class  $h$ . We will also introduce the notion of a locally minimal orbit, if the associated closed curve minimize the length, over all closed curves in its neighborhood and with homology  $h$ .

We will prove that for a generic  $H^s$ , for energies  $E_0 \leq E \leq \bar{E}$ , the globally minimal orbits are hyperbolic. To achieve this, we study generic properties of *non-degenerate* orbits. We say that a periodic orbit of a Hamiltonian system is non-degenerate, if the differential of the associated Poincaré return map on its energy surface does not have 1 as an eigenvalue. Note that a non-degenerate orbit could have eigenvalues on the unit circle, hence not necessarily hyperbolic.

The proof of Theorem 4 consists of three parts.

1. In section A.1, we prove a Kupka-Smale-like theorem about non-degeneracy of periodic orbits. For a fixed energy surface, generically, all periodic orbits are non-degenerate. This fails for an interval of energies. We show that while degenerate periodic orbits exists, there are only finitely many of them. Moreover, there could be only a particular type of bifurcation for any family of periodic orbits crossing a degeneracy.
2. In section A.2, we show that a non-degenerate locally minimal orbit is always *hyperbolic*. Using part I, we show that for each energy, the globally minimal orbits is chosen f a finite family of *hyperbolic* locally minimal orbits.
3. In section A.3, we finish the proof of Theorem 4. This amounts to proving the finite local families obtained from part II are “in general position”.

### A.1 Generic properties of periodic orbits

We simplify notations and drop the superscript “s” from the notation of the slow mechanical system. Moreover, we treat  $U$  as a parameter, and write

$$H^U(\varphi, I) = K(I) - U(\varphi), \quad \theta \in \mathbb{T}^2, \quad I \in \mathbb{R}^2, \quad U \in C^r(\mathbb{T}^2). \quad (45)$$

We shall use  $U$  as *an infinite-dimensional parameter*. As before  $K$  is a kinetic energy and it is fixed. Denote by  $\mathcal{G}^r = C^r(\mathbb{T}^2)$  the space of potentials,  $x$  denotes  $(\varphi, I)$ ,

$W = \mathbb{T}^2 \times \mathbb{R}^2$ , and either  $\varphi_t^U$  or  $\Phi(\cdot, t, U)$  denotes the flow of (45). We will use  $\chi^U(x) = (\partial K, \partial U)(x)$  to denote the Hamiltonian vector field of  $H^U$  and use  $S_E$  to denote the energy surface  $\{H^U = E\}$ . We may drop the superscript  $U$  when there is no confusion.

By the invariance of the energy surface, the differential map  $D_x \varphi_t^U$  defines a map

$$D_x \varphi_t^U(x) : T_x S_{H(x)} \longrightarrow T_{\varphi_t^U(x)} S_{H(x)}.$$

Since the vector field  $\chi(x)$  is invariant under the flow,  $D_x \varphi$  induces a factor map

$$\bar{D}_x \varphi_t^U(x) : T_x S_{H(x)} / \mathbb{R} \chi(x) \longrightarrow T_{\varphi_t^U(x)} S_{H(x)} / \mathbb{R} \chi(\varphi_t(x)).$$

Given  $U_0 \in \mathcal{G}^r$ ,  $x_0 \in W$  and  $t_0 \in \mathbb{R}$ , let

$$\mathcal{V} = V(x_0) \times (t_0 - a, t_0 + a) \times V(U_0)$$

be a neighborhood of  $(x_0, t_0, U_0)$ ,  $V(x_0, t_0)$  of  $(x_0, t_0)$ , and  $V(\varphi_{t_0}^{U_0}(x_0))$  a neighborhood of  $\varphi_{t_0}^{U_0}(x_0)$ , such that

$$\varphi_t^U(x) \in V(\varphi_{t_0}^{U_0}(x_0)), \quad (x, t, U) \in \mathcal{V}.$$

By fixing the local coordinates on  $V(x_0)$  and  $V(\varphi_{t_0}^U(x_x))$ , we define

$$\tilde{D}_x \Phi : \mathcal{V} \longrightarrow Sp(1),$$

where  $\tilde{D}_x \Phi(x, t, U)$  is the  $2 \times 2$  symplectic matrix associated to  $\bar{D}_x \varphi_t^U(x)$  under the local coordinates. The definition depends on the choice of coordinates.

Let  $\{\varphi_t^{U_0}(x_0)\}$  be a periodic orbit with minimal period  $t_0$ . The periodic orbit is *non-degenerate* if and only if 1 is *not* an eigenvalue of  $\tilde{D}_x \Phi(x_0, t_0, U_0)$ <sup>27</sup>. Furthermore, we identify two types of degeneracies:

1. A degenerate periodic orbit  $(x_0, t_0, U_0)$  is of *type I* if  $\tilde{D}_x \Phi(x_0, t_0, U_0) = Id$ , the identity matrix;
2. It is of *type II* if  $\tilde{D}_x \Phi(x_0, t_0, U_0)$  is conjugate to the matrix  $[1, \mu; 0, 1]$  for  $\mu \neq 0$ .

Denote

$$N = \left\{ \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} : \mu \in \mathbb{R} \setminus \{0\} \right\}, \quad \mathcal{O}(N) = \{BAB^{-1} : A \in N, B \in Sp(1)\}.$$

Then  $(x_0, t_0, U_0)$  is of type II if and only if  $\tilde{D}_x \Phi(x_0, t_0, U_0) \in \mathcal{O}(N)$ .

---

<sup>27</sup>Note that we are interested in non-degeneracy for minimal period of periodic orbits only. As the result eigenvalues given by  $\exp(2\pi i p/q)$  with integer  $p, q \neq 0$  are allowed

**Lemma A.1.** *The set  $\mathcal{O}(N)$  is a 2-dimensional submanifold of  $Sp(1)$ .*

*Proof.* Any matrix in  $\mathcal{O}(N)$  can be expressed by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 - ac\mu & a^2\mu \\ -c^2\mu & 1 - ac\mu \end{bmatrix},$$

where  $ad - bc = 1$  and  $\mu \neq 0$ . Write  $\alpha = a^2\mu$  and  $\beta = ac\mu$ , we can express any matrix in  $\mathcal{O}(N)$  by

$$\begin{bmatrix} 1 - \beta & \alpha \\ 1 - \beta^2/\alpha & 1 - \beta \end{bmatrix}. \quad (46)$$

□

The standard Kupka-Smale theorem (see [66], [67]) *no longer holds for an interval of energies*. Generically, periodic orbits appear in one-parameter families and may contain degenerate ones. However, while degenerate periodic orbits may appear, generically, a family of periodic orbits crosses the degeneracy “transversally”. This is made precise in the following theorem.

**Theorem 25.** *There exists residue subset of potentials  $\mathcal{G}'$  of  $\mathcal{G}^r$ , such that for all  $U \in \mathcal{G}'$ , the following hold:*

1. *The set of periodic orbits for  $\varphi_t^U$  form a submanifold of dimension 2. Since a periodic orbit itself is a 1-dimensional manifold, distinct periodic orbits form one-parameter families.*
2. *There is no degenerate periodic orbits of type I.*
3. *The set of periodic orbits of type II form a 1-dimensional manifold. Factoring out the flow direction, the set of type II degenerate orbits are isolated.*
4. *For  $U_0 \in \mathcal{G}^r$ , let  $\Lambda^{U_0} \subset W \times \mathbb{R}^+$  denote the set of periodic orbits for  $\varphi_t^U$ , and  $\Lambda_N^{U_0} \subset \Lambda^{U_0}$  denote the set of type II degenerate ones. Then for any  $(x_0, t_0) \in \Lambda_N^{U_0}$ , let  $V(x_0, t_0)$  be a neighborhood of  $(x_0, t_0)$ . Then*

$$\tilde{D}_x \Phi|_{U=U_0} : \Lambda^{U_0} \cap V(x_0, t_0) \longrightarrow Sp(1)$$

*is transversal to  $\mathcal{O}(N) \subset Sp(1)$ .*

**Remark A.1.** *Statement 4 of the theorem can be interpreted in the following way. Let  $A(\lambda)$  be the differential of the Poincare return map on associated with a family of periodic orbits. Then if  $A(\lambda_0) \in \mathcal{O}(N)$ , then the tangent vector  $A'(\lambda_0)$  is transversal to  $\mathcal{O}(N)$ .*

We can improve the set  $\mathcal{G}'$  to an open and dense set, if there is a lower and upper bound on the minimal period.

**Corollary A.2.** 1. *Given  $0 < T_0 < T_1$ , there exists an open and dense subset  $\mathcal{G}' \subset \mathcal{G}^r$ , such that the set of periodic orbits with minimal period in  $[T_0, T_1]$  satisfies the conclusions of Theorem 25.*

2. *For any  $U_0 \in \mathcal{G}'$ , there are at most finitely many type II degenerate periodic orbits. Moreover, there exists a neighbourhood  $V(U_0)$  of  $U_0$ , such that the set of type II degenerate periodic orbits depends smoothly on  $U$ . (This means the number of such periodic orbits is constant on  $V(U_0)$ , and each periodic orbit depends smoothly on  $U$ .)*

We define

$$F : W \times \mathbb{R}^+ \times \mathcal{G}^r \longrightarrow W \times W, \quad (47)$$

$$F(x, t, U) = (x, \Phi(x, t, U)).$$

$F$  is a  $C^{r-1}$ -map of Banach manifolds. Define the diagonal set by  $\Delta = \{(x, x)\} \subset W \times W$ . Then  $\{\varphi_t^{U_0}(x_0)\}$  is an period orbit of period  $t_0$  if and only if  $(x_0, t_0, U_0) \in F^{-1}\Delta$ .

**Proposition A.3.** *Assume that  $(x_0, t_0, U_0) \in F^{-1}\Delta$  or, equivalently,  $x_0$  is periodic orbit of period  $t_0$  for  $H^U$  and that  $t_0$  is the minimal period, then there exists a neighborhood  $\mathcal{V}$  of  $(x_0, t_0, U_0)$  such that the map*

$$d\pi_{\Delta}^{\perp} D_{(x,t,U)} F : T_{(x,t,U)}(W \times \mathbb{R}^+ \times \mathcal{G}^r) \longrightarrow T_{F(x,t,U)}(W \times W)/T\Delta$$

*has co-rank 1 for each  $(x, t, U) \in \mathcal{V}$ , where  $d\pi_{\Delta}^{\perp} T(W \times W) \longrightarrow T(W \times W)/T\Delta$  is the standard projection along  $T\Delta$ .*

**Remark A.2.** *If the aforementioned map has full rank, then the map is called transversal to  $\Delta$  at  $(x_0, t_0, U_0)$ . However, the transversality condition fails for our map.*

Given  $\delta U \in \mathcal{G}^r$ , the directional derivative  $D_U \Phi \cdot \delta U$  is defined as follows  $\frac{\partial}{\partial \epsilon}|_{\epsilon=0} \Phi(x, t, U + \epsilon \delta U)$ . The differential  $D_U \Phi$  then defines a map from  $T\mathcal{G}^r$  to  $T_{\Phi(x,t,U)}W$ . The following hold for this differential map:

**Lemma A.4.** [66] *Assume that there exists  $\epsilon > 0$  such that the orbit of  $x$  has no self-intersection for the time interval  $(\epsilon, \tau - \epsilon)$ , then the map*

$$D_U \Phi(x, \tau, U) : \mathcal{G}^r \longrightarrow T_{\Phi(x,\tau,U)}W$$

*generates a subspace orthogonal to the gradient  $\nabla H^U(\Phi(x, \tau, U))$  and the Hamiltonian vector field  $\chi^U(\Phi(x, \tau, U))$  of  $H^U$ .*

*Proof.* We refer to [66], Lemma 16 and 17. We note that while the proof was written for a periodic orbit of minimal period  $\tau$ , the proof holds for non-self-intersecting orbit.  $\square$

*Proof of Proposition A.3.* We note that if  $\{\varphi_t^U(x)\}$  is a periodic orbit of minimal period  $\tau$ , then the orbit  $\{\varphi_t^{U'}(x')\}$  satisfies the assumptions of Lemma A.4. It follows that the matrix

$$d\pi_\Delta^\perp \circ D_U F = \begin{bmatrix} D_x \Phi - I & D_t \Phi & D_U \Phi \end{bmatrix}$$

has co-rank 1, since the last two component generates the subspace

$$\text{Image}(D_U \Phi) + \mathbb{R}\chi^U,$$

which is a subspace complementary to  $\nabla H^U$ .  $\square$

Proposition A.3 allows us to apply the constant rank theorem in Banach spaces.

**Proposition A.5.** *The set  $F^{-1}\Delta$  as a subset of a Banach space is a submanifold of codimension  $2n - 1$ . If  $r \geq 4$ , then for generic  $U \in \mathcal{G}^r$ ,  $F^{-1}\Delta \cap \pi_U^{-1}\{U\}$  is a 2-dimensional manifold.*

*Proof.* We note that the kernel and cokernel of the map  $d\pi \circ D_U F$  has finite codimension, hence the constant rank theorem (see [1], Theorem 2.5.15) applies. As a consequence, we may assume that locally,  $\Delta = \Delta_1 \times (-a, a)$  and that the map  $\pi_1 \circ F$  has full rank. Since the dimension of  $\Delta_1$  is  $2n - 1$ ,  $F^{-1}\Delta$  is a submanifold of codimension  $2n - 1$ . The second claim follows from Sard's theorem.  $\square$

Denote  $\Lambda = F^{-1}\Delta$ . On a neighbourhood  $\mathcal{V}$  of each  $(x_0, t_0, U_0) \in \Lambda$ , we define the map

$$\tilde{D}_x \Phi : \Lambda \cap \mathcal{V} \longrightarrow Sp(1), \quad \tilde{D}_x \Phi(x, t, U) = \tilde{D}\varphi_t^U(x). \quad (48)$$

First we refer to the following lemma of Oliveira:

**Lemma A.6** ([66], Theorem 18). *For each  $(x_0, t_0, U_0) \in \Lambda$  such that  $t_0$  is the minimal period, let  $\tilde{\mathcal{G}}$  be the set of tangent vectors in  $T_{(x_0, t_0, U_0)}\Lambda$  of the form  $(0, 0, V)$ . Then the map*

$$D_U \tilde{D}_x \Phi : \tilde{\mathcal{G}} \longrightarrow T_{\tilde{D}_x \Phi(x_0, t_0, U_0)} Sp(1)$$

*has full rank.*

**Corollary A.7.** *The map (48) is transversal to the submanifold  $\{Id\}$  and  $\mathcal{O}(N)$  of  $Sp(1)$ .*

Denote

$$\Lambda_{Id} = \Lambda \cap \tilde{D}_x \Phi^{-1}(\{Id\}) \text{ and } \Lambda_N = \Lambda \cap \tilde{D}_x \Phi^{-1}(\mathcal{O}(N)).$$

Note that the expression is well defined because both preimages are defined independent of local coordinate changes.

*Proof of Theorem 25.* The first statement of the theorem follows from Proposition A.5.

As the subset  $\{Id\}$  has codimension 3 in  $Sp(1)$ ,  $\Lambda_{Id}$  has codimension 3 in  $\Lambda$ , and hence has codimension  $2n+2$  in  $W \times \mathbb{R}^+ \times \mathcal{G}^r$ . By Sard's lemma, for a generic  $U \in \mathcal{G}^r$ , the set  $\Lambda_{Id} \cap \pi_U^{-1}$  is empty. This proves the second statement of the theorem.

Since the set  $\mathcal{O}(N)$  has codimension 1,  $\Lambda_N$  has codimension 1 in  $\Lambda$ , and hence has codimension  $2n$  in  $W \times \mathbb{R}^+ \times \mathcal{G}^r$ . As a consequence, generically, the set  $\Lambda_N^U = \Lambda_N \cap \pi_U^{-1}$  has dimension 1. This proves the third statement.

Fix  $U_0 \in \mathcal{G}'$ , the set  $\Lambda^{U_0} = \Lambda \cap \pi_{U_0}^{-1}(U_0)$  has dimension 2, while  $\Lambda_N^{U_0} \subset \Lambda^{U_0}$  has dimension 1. It follows that at any  $(x_0, t_0) \in \Lambda_N^{U_0}$ , there exists a tangent vector

$$(\delta x, \delta t) \in T_{(x_0, t_0)} \Lambda^{U_0} \setminus T_{(x_0, t_0)} \Lambda_N^{U_0}$$

such that

$$(\delta x, \delta t, 0) \in T_{(x_0, t_0, U_0)} \Lambda \setminus T_{(x_0, t_0, U_0)} \Lambda_N.$$

It follows that

$$D_{(x,t)} \tilde{D}_x \Phi|_{U=U_0}(x_0, t_0) = D_{(x,t,U)} \tilde{D}_x \Phi(x_0, t_0, U_0) \cdot (\delta x, \delta t, 0)$$

is not tangent to  $\mathcal{O}(N)$ . Since  $\mathcal{O}(N)$  has codimension 1, this implies that the map  $\tilde{D}_x \Phi|_{U=U_0}$  is transversal to  $\mathcal{O}(N)$ . This proves the fourth statement.  $\square$

*Proof of Corollary A.2.* If a potential  $U \in \mathcal{G}'$ , then by Theorem 25 conditions 1–4 are satisfied. In particular, all periodic orbits are either non-degenerate or degenerate satisfying conditions 3 and 4. Non-degenerate periodic orbits of period bounded both from zero and infinity form a compact set. Therefore, they stay non-degenerate for all potential  $C^r$ -close to  $U$ . By condition 3 degenerate periodic orbits are isolated. This implies that there are finitely many of them. Condition 4 is a transversality condition, which is  $C^r$  open for each degenerate orbit.  $\square$

Fix  $U \in \mathcal{G}'$  as in Corollary A.2. It follows that periodic orbits of  $\varphi_t^U$  for one-parameter families. We now discuss the generic bifurcation of such a family at a degenerate periodic or

**Proposition A.8.** *Let  $(x_\lambda, t_\lambda)$  be a family of periodic orbits such that  $(x_0, \lambda_0)$  is degenerate. The one side of  $\lambda = \lambda_0$ , the matrix  $\tilde{D}_x \varphi_{t_\lambda}^U(x_\lambda)$  has a pair of distinct real eigenvalues; on the other side of  $\lambda = \lambda_0$ , it has a pair of complex eigenvalues.*



*Proof.* Write  $A(\lambda) = \tilde{D}_x \varphi_{t_\lambda}^U(x_\lambda)$  for short. By choosing a proper local coordin we may assume that  $A(\lambda_0) = [1, \mu; 0, 1]$ . The tangent space to  $Sp(1)$  at  $[1, \mu; 0, 1]$  is given by the set of traceless matrices  $[a, b; c, -a]$ . Using (46), we have a basis of the tang space to  $\mathcal{O}(N)$  to  $[1, \mu; 0, 1]$  is given by

$$\begin{bmatrix} 0 & 1 \\ -\beta^2/\alpha^2 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ -2\beta/\alpha & -1 \end{bmatrix}.$$

An orthogonal matrix to this space, using the inner product  $tr(A^T B)$ , is given by  $[0, 0; 1, 0]$ . As a consequence, a matrix  $[a, b; c, -a]$  is transversal to  $\mathcal{O}(N)$  if and only if  $c \neq 0$ .

The eigenvalues of the matrix

$$\begin{bmatrix} 1 + ah & \mu + bh \\ ch & 1 - ah \end{bmatrix}$$

are given by  $\lambda = 1 \pm \sqrt{a^2 h^2 - bch^2 - \mu ch}$ . Using  $\mu \neq 0$  and  $c \neq 0$  we obtain that  $a^2 h^2 - bch^2 - \mu ch$  changes sign at  $h = 0$ . This proves our proposition.  $\square$

## A.2 Generic properties of minimal orbits

In this section we study the second item of the plan proposed in the beginning of this Appendix. Namely, we analyze properties of families of minimal orbits. It turns out that non-degenerate minimizers are hyperbolic. Naturally, hyperbolic periodic orbits form smooth families parametrized by energy. However, generically there are not only non-degenerate local minimizers, but also isolated degenerate ones, which is somewhat surprising (see Proposition A.14). We manage to show that such degenerate local minimizers generically are not global. The main result of the section is Theorem 26.

Let  $d_E$  denote the metric derived from the Riemannian metric  $\rho_E$ . We define the arclength of any continuous curve  $\gamma : [t, s] \longrightarrow \mathbb{T}^2$  by

$$l_E(\gamma) = \sup \sum_{i=0}^{N-1} d_E(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions  $\{[t_i, t_{i+1}]\}_{i=0}^{N-1}$  of  $[t, s]$ . A curve  $\gamma$  is called rectifiable if  $l_E(\gamma)$  is finite.

A curve  $\gamma : [a, b] \longrightarrow \mathbb{T}^2$  is called piecewise regular, if it is piecewise  $C^1$  and  $\dot{\gamma}(t) \neq 0$  for all  $t \in [a, b]$ . A piecewise regular curve is always rectifiable.

We write

$$l_E(h) = \inf_{\eta \in C_h^E} l_E(\eta),$$

where  $\mathcal{C}_h^E$  denote the set of all piecewise regular closed curves with homology  $h \in H_1(\mathbb{T}^2, \mathbb{Z})$ . A curve realizing the infimum is the shortest geodesic curve in the homology  $h$ , which we will also refer to as a global  $(\rho_E, h)$ -minimizer.

We fix the homology  $h$  and will omit  $h$  when there is no confusion.

It is well known that for any  $E > -\min_\varphi U(\varphi)$ , a global  $\rho_E$ -minimizer is a closed  $\rho_E$ -geodesic. Hence, it corresponds to a periodic orbit of the Hamiltonian flow.

A global  $\rho_E$ -minimizer is always a closed geodesic of the Riemannian metric  $\rho_E$ , and hence is associated with a periodic orbit of the Hamiltonian flow. We say that a closed geodesic  $\gamma$  of the  $\rho_E$  is hyperbolic if the associated Hamiltonian orbit  $\tilde{\gamma}$  is hyperbolic. We have the following statement, in relation to the discussions in section A.1.

**Proposition A.9.** *Assume that  $\gamma$  is a  $(\rho_E, h)$ -minimizer, and assume that the associated periodic orbit  $\tilde{\gamma}$  is nondegenerate. Then  $\gamma$  is hyperbolic.*

We first introduce some definitions. Let  $L_H(\varphi, v)$  be the Lagrangian associated to the Hamiltonian  $H(\varphi, I)$ , and we use the same notation to denote its lift to the universal cover  $\mathbb{R}^2 \times \mathbb{R}^2$ . A piecewise  $C^1$  curve  $x : \mathbb{R} \rightarrow \mathbb{R}^2$  is called  $L_H$ -minimizing, if

$$\int_a^b L_H(x(t), \dot{x}(t)) dt = \inf_y \int_a^b L_H(y, \dot{y}(t)) dt,$$

where the infimum is taken over all piecewise  $C^1$  curves  $y : [a, b] \rightarrow \mathbb{R}^2$  with  $y(a) = x(a)$  and  $y(b) = x(b)$ . It is well known that any  $L_H$ -minimizing curve is a solution of the Euler-Lagrange equation, and corresponds to a solution of the Hamiltonian equation.

Let  $\tilde{\gamma}(t) = (x(t), p(t))$  be a solution of the Hamiltonian equation. We use  $(\delta x, \delta p)$  to denote the local coordinates of  $T(\mathbb{T}^2 \times \mathbb{R}^2)$  induced by the coordinates  $(x, p)$ . We call the linear subspace  $V(x, p) = \{(0, \delta p)\} \subset T_{(x, p)}(\mathbb{T}^2 \times \mathbb{R}^2)$  the vertical subspace. The orbit  $\tilde{\gamma}$  is called *disconjugate*, if

$$D\varphi_t(x(s), p(s))V(x(s), p(s)) \cap V(x(s+t), p(s+t)) = \{0\},$$

for any  $s \in \mathbb{R}$  and  $t > 0$ .

**Lemma A.10.** *1. If  $\gamma$  is a  $(\rho_E, h)$ -minimizer, then the associated Hamiltonian orbit  $\tilde{\gamma}$  lift to an  $L_H$ -minimizer.*

*2. If an orbit  $(x(t), p(t))$  is an  $L_H$ -minimizer, then it is disconjugate.*

*3. If an orbit  $(x(t), p(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$  is disconjugate, then the differential map  $D\varphi_t$  admits a 2-dimensional invariant bundle contained in  $TS_{H(x(t), p(t))}$ .*

The statements in Lemma A.10 are now classical, so we will only point to some references. It follows from a theorem of Diaz Carneiro [28] that a  $(\rho_E, h)$ –minimizer corresponds to a Mather minimal measure with rotation number  $h$ . All orbits contained in the support of a Mather minimal measures are  $L_H$ –minimal (see [53], for example). For the second statement, we refer to Contreras and Iturriaga ([27], Corollary 4.2). The invariant bundle in the third statement is one of the Green bundles, for the Hamiltonian version, we refer to [27], Proposition A.

*Proof of Proposition A.9.* Assume that  $\gamma$  is a  $(\rho_E, h)$ –minimizer, and its lift  $\tilde{\gamma}(t) = (\varphi(t), p(t))$  is a nondegenerate periodic orbit of period  $T$ . We have that either  $\tilde{\gamma}$  is hyperbolic, or the matrix  $\tilde{D}\varphi_T(\varphi(0), p(0))$ , defined in section A.1, has complex eigenvalues. As a consequence,  $\tilde{D}\varphi_T(\varphi(0), p(0))$  admits no one-dimensional invariant subspace. Recall that the map  $\tilde{D}\varphi_t$  is the restriction of  $D\varphi_t$  to the energy surface, with the flow direction quotient out. Hence, the map  $D\varphi_T(\varphi(0), p(0))$ , restricted to the energy surface, admits no 2-dimensional invariant bundles. This is a contradiction with Lemma A.10.  $\square$

We recall the notion of local  $\rho_E$ –minimizers. Given an open set  $V \subset \mathbb{T}^2$ , a continuous closed curve  $\gamma \subset V$  with homology  $h$  is a  $(\rho_E, h, V)$ –minimizer, if

$$l_E(\gamma) = \inf_{\eta \in \mathcal{C}_h^E, \eta \subset \bar{V}} l_E(\eta), \quad (49)$$

where  $\mathcal{C}_h^E$  denote the set of all piecewise regular closed curves with homology  $h$ . ( $\gamma$  is necessarily rectifiable.) A curve  $\gamma$  is a local  $(\rho_E, h)$ –minimizer, if there exists an open set  $V \supset \gamma$  such that  $\gamma$  is a  $(\rho_E, h, V)$ –minimizer. By Lemma A.11 below, local minimizers are also closed geodesics, and hence corresponds to a Hamiltonian orbit.

The main goal of this section is to prove that, generically, each (global)  $\rho_E$ –minimizer is chosen among finitely many hyperbolic local minimizers.

**Theorem 26.** *Given  $0 < E_0 < \bar{E}$ , there exists an open and dense subset  $\mathcal{G}'$  of  $\mathcal{G}^r$ , such that for each  $U \in \mathcal{G}'$ , the Hamiltonian  $H(\varphi, I) = K(I) - U(\varphi) + \min_{\varphi} U(\varphi)$  satisfies the following statements. There exists finitely many smooth families of local minimizers*

$$\xi_j^E, \quad a_j - \sigma \leq E \leq b_j + \sigma, \quad j = 1, \dots, N,$$

and  $\sigma > 0$ , with the following properties.

1. All  $\xi_j^E$  for  $a_j - \sigma \leq E \leq b_j + \sigma$  are hyperbolic.
2.  $\bigcup_j [a_j, b_j] \supset [E_0, \bar{E}]$ .
3. For each  $E_0 \leq E \leq \bar{E}$ , any global minimizer is contained in the set of all  $\xi_j^E$ 's such that  $E \in [a_j, b_j]$ .

Proof of Theorem 26, occupies the rest of this section.

**Lemma A.11.** 1. If the set  $\{\eta \in \mathcal{C}_h^E; \eta \subset \bar{V}\}$  is nonempty, then the infimum in (49) can be achieved at a rectifiable curve.

2. Any rectifiable local  $\rho_E$ -minimizer is a closed geodesic of the Riemannian metric  $\rho_E$ .

*Proof.* *Statement 1.* Let  $\eta_n \in \mathcal{C}_h^E$ ,  $\eta \subset \bar{V}$  be a sequence of piecewise regular curves with

$$\lim_{n \rightarrow \infty} l_E(\eta_n) = \inf_{\eta \in \mathcal{C}_h^E, \eta \subset \bar{U}} l_E(\eta).$$

Assume that all  $\eta_n$  are parametrized on  $[t, s]$  with uniformly bounded derivatives. This is possible because all  $\eta_n$  has uniformly bounded length. Then by passing to a subsequence, we may assume that  $\eta_n$  converges uniformly to  $\eta_*$ . It suffices to show that

$$l_E(\eta_*) \leq \lim_{n \rightarrow \infty} l_E(\eta_n).$$

Take any finite partition  $\{[t_i, t_{i+1}]\}_{i=0}^{N-1}$  of  $[t, s]$ , we have

$$d_E(\eta_*(t_i), \eta_*(t_{i+1})) = \lim_{n \rightarrow \infty} d_E(\eta_n(t_i), \eta_n(t_{i+1})) \leq \limsup_{n \rightarrow \infty} l_E(\eta_n|_{[t_i, t_{i+1}]}) ,$$

hence

$$\sum_{i=0}^{N-1} d_E(\eta_*(t_i), \eta_*(t_{i+1})) \leq \limsup_{n \rightarrow \infty} \sum_{i=0}^{N-1} l_E(\eta_n|_{[t_i, t_{i+1}]}) = \limsup_{n \rightarrow \infty} l_E(\eta_n).$$

Taking supremum over all partitions, we obtain statement 1.

*Statement 2.* Let  $\delta > 0$  be such that any two points with the  $d_E$ -distance less than  $\delta$  can be connected by a unique geodesic realizing the distance.

For a rectifiable local  $\rho_E$ -minimizer  $\gamma : [t, s] \rightarrow \mathbb{T}^2$ , define  $\delta_0 = \min\{\delta, d_E(\gamma, \partial U)\}$ . Let  $\{[t_i, t_{i+1}]\}_{i=0}^{N-1}$  be a partition  $[t, s]$  with  $l_E(\gamma|_{[t_i, t_{i+1}]}) < \delta$ . Then  $\gamma(t_i)$  and  $\gamma(t_{i+1})$  can be connected by a geodesic  $\xi_i$  contained in  $U$  with the same arc-length as  $\gamma|_{[t_i, t_{i+1}]}$ .

It follows that  $\xi := \xi_0 * \cdots * \xi_{N-1}$  is also a local minimizer. Using the standard arguments of Riemannian geometry, we conclude that  $\xi_i$  and  $\xi_{i+1}$  must have matching unit tangent vector, and hence  $\xi$  itself is a geodesic.

We obtained a geodesic  $\xi$  which coincide with  $\gamma$  at the points  $\gamma(t_i)$ . Since the argument works with arbitrary refinement of the partition, we conclude that  $\gamma$  coincide with  $\xi$ .  $\square$

Any hyperbolic orbit is locally unique on the energy surface, and extends to a one parameter family of hyperbolic orbits.

**Lemma A.12.** *Assume that  $\tilde{\gamma}_{E_0}$  is a hyperbolic periodic orbit in the energy surface  $S_{E_0}$ . Then the following hold:*

1. *There exists a neighbourhood  $\tilde{V}$  of  $\tilde{\gamma}$  in  $S_{E_0}$ , such that  $\tilde{\gamma}$  is the unique periodic orbit in this neighbourhood.*
2. *There exists  $\delta > 0$  and a neighbourhood  $\tilde{V}$  of  $\tilde{\gamma}_{E_0}$  such that for any Hamiltonian  $H'$  that is  $\delta$ -close to  $H$  in the  $C^2$  norm, and  $|E' - E_0| \leq \delta$ , there exists a unique hyperbolic periodic orbit  $\tilde{\gamma}'$  of  $H'$  in  $\tilde{V}$  with energy  $E'$ .*
3. *There exists  $\delta > 0$  and a smooth family  $\tilde{\gamma}_E \subset \tilde{V}$ ,  $E_0 - \delta \leq E \leq E_0 + \delta$ , each of them hyperbolic, and is unique on  $\tilde{V}$ .*
4. *Any smooth one-parameter family of hyperbolic periodic orbit is monotone in energy.*

*Proof.* Choose a transversal section to  $\tilde{\gamma}(0)$ , and define a Poincare return map  $\Phi_{E_0}$  on this section. A periodic orbit corresponds to a fixed point of the Poincare return map. The first three statements of this lemma follow directly from the inverse function theorem.

We now prove the fourth statement. Assume that  $\tilde{\gamma}_\lambda$  is a family of hyperbolic periodic orbits, and the function  $H(\tilde{\gamma}_\lambda(0))$  is not monotone. Assume, by contradiction, that  $\lambda_0$  is a local minimum, with  $E_0 = H(\tilde{\gamma}_{\lambda_0}(0))$ . Then for small  $E > E_0$ , there exists at least two periodic orbits  $\gamma_{\lambda_1(E)}$  and  $\gamma_{\lambda_2(E)}$ . However, this contradicts with statement 2. We can similarly rule out local maxima.  $\square$

For hyperbolic local minimizers, we have the following local description.

**Lemma A.13.** *Assume that  $\gamma_{E_0}$  is a hyperbolic local  $\rho_{E_0}$ -minimizer. The following hold.*

1. *There exists a neighbourhood  $V$  of  $\gamma_{E_0}$ , such that  $\gamma_{E_0}$  is the unique local  $\rho_{E_0}$ -minimizer on  $V$ .*
2. *There exists  $\delta > 0$  such that for any  $U' \in C^r(\mathbb{T}^2)$  with  $\|U - U'\|_{C^2} \leq \delta$  and  $|E' - E_0| \leq \delta$ , the Hamiltonian  $H'(\varphi, I) = K(I) - U'(\varphi)$  admits a hyperbolic local minimizer in  $V$ .*
3. *There exists  $\delta > 0$  and a smooth family  $\gamma_E \subset V$ ,  $E_0 - \delta \leq E \leq E_0 + \delta$ , such that each of them is a hyperbolic local minimizer.*

*Proof. Statement 1.* Assume, by contradiction, that there is a sequence of local minimizers  $\eta_n$  accumulating to  $\gamma_{E_0}$ . By Lemma A.11, statement 2, each local minimizer is a closed geodesic, and hence corresponds to a periodic orbit. Let  $\tilde{\eta}_n$  and  $\tilde{\gamma}_{E_0}$

be the associated Hamiltonian orbit. We have that  $\tilde{\eta}_n$  converges to  $\tilde{\gamma}_{E_0}$  in the phase space but this contradicts with Lemma A.12, statement 1.

*Statement 2.* We will show that the depth of the minimum is nondegenerate. More precisely, we show there exists a neighbourhood  $V$  of  $\gamma_{E_0}$ , such that

$$\inf_{\eta \subset \tilde{V}, \eta \cap \partial V \neq \emptyset, \eta \in \mathcal{C}_h^{E_0}} l_{E_0}(\eta) > \inf_{\eta \subset \tilde{V}, \eta \in \mathcal{C}_h^{E_0}} l_{E_0}(\eta). \quad (50)$$

Assume, by contradiction, that there exists a sequence of shrinking neighbourhoods  $V_n$ , such that (50) is an equality for each  $V_n$ . By an identical argument to the proof of Lemma A.11, statement 1, we conclude that the infimum in the left hand side of (50) can be achieved at a rectifiable curve  $\xi_n$ , not identical to  $\gamma_{E_0}$ , each  $n$ . Each  $\xi_n$  is a local minimizer. This contradicts with uniqueness obtained from statement 1.

*Statement 3.* We note that (50) persists under small perturbation of the metric conclude that for  $|E - E_0| \leq \delta$ , the metric  $\rho_E$  admits a local minimizer in  $V$ , where  $V$  is the neighborhood from statement 2. By choosing  $V$  and  $\delta$  smaller if necessary, we can make sure the associated periodic orbits  $\tilde{\gamma}_E$  are contained in  $\tilde{V}$ , where  $\tilde{V}$  is the neighborhood in Lemma A.12, statement 3. Uniqueness then imply that the family  $\tilde{\gamma}_E$  coincide with the family obtained in Lemma A.12, statement 3.  $\square$

We now use the information obtained to classify the set of global minimizers.

- Consider the Hamiltonian  $H(\varphi, I) = K(I) - U(\varphi) + \min_{\varphi} U(\varphi)$ . For  $0 < E_0 < \bar{E}$ , it is easy to see that any periodic orbit in the energy  $E_0 \leq E \leq \bar{E}$  has a lower bound and upper bound on the minimal period, which depends only on  $E_0$  and  $\bar{E}$ . Hence, Corollary A.2 applies.
- By Corollary A.2, generically, there are at most finitely many degenerate global minimizers, the rest are nondegenerate (as periodic orbits). By Proposition A.9, they must hyperbolic.
- Since a global minimizer is always a local minimizer, using Lemma A.13, it extends to a smooth one parameter family of local minimizers. The extension can be continued until the family accumulates to a degenerate minimizer. This family can no longer be extended as potential global minimizers – by Proposition A.8, it is accumulated by periodic orbits of complex eigenvalues.
- It is well known that for a fixed energy, any two global minimizers do not cross (see for example, [51]). We assume that the local extension of these global minimizers also do not cross, for a fixed energy.
- There are at most finitely many families of local minimizers, because they are isolated (Lemma A.13, statement 1), and do not accumulate (Lemma A.12, statement 4).

- We haven't excluded the case that a global minimizer is taken at an isolated degenerate periodic orbit. While by Proposition A.8, it must be accumulated by hyperbolic orbits, these hyperbolic orbit may not be minimizers.

We have proved the following statement.

**Proposition A.14.** *Given  $0 < E_0 < \bar{E}$ , there exists an open and dense subset  $\mathcal{G}'$  of  $\mathcal{G}^r$ , such that for each  $U \in \mathcal{G}'$ , for the Hamiltonian  $H(\varphi, I) = K(I) - U(\varphi) + \min_{\varphi} U(\varphi)$ , such that the following hold.*

1. *There are at most finitely many (maybe none) isolated global minimizers  $\xi_j^{c,j,d}$ , that are degenerate.*
2. *There are finitely many smooth families of local minimizers*

$$\bar{\xi}_j^E, \quad \bar{a}_j \leq E \leq \bar{b}_j, \quad j = 1, \dots, N,$$

*with  $[E_0, \bar{E}] \supset \bigcup [\bar{a}_j, \bar{b}_j]$ , such that  $\bar{\xi}_j^E$  are hyperbolic for  $\bar{a}_j < E < \bar{b}_j$ . The set  $\bar{\xi}_j^E$  for  $E = \bar{a}_j, \bar{b}_j$  may be hyperbolic or degenerate.*

3. *For a fixed energy surface  $E$ , the sets  $\{\xi_j^{E,d}\}$  and  $\bigcup_{\bar{a}_j \leq E \leq \bar{b}_j} \xi_j^E$  are pairwise disjoint.*
4. *For each  $E_0 \leq E \leq \bar{E}$ , the global minimizer is chosen among the set of all  $\xi_j^{c,j,d}$  with  $E = c_j$ , or one of the local minimizers  $\bar{\xi}_j^E$  with  $E \in [\bar{a}_j, \bar{b}_j]$ .*

*Proof of Theorem 26.* We first show that the set of potentials  $U$  satisfying the conclusion of Theorem 26 is open. By Lemma A.13, the family of local minimizers persists under small perturbation of the potential  $U$ . It suffices to show that for sufficiently small perturbation of  $U$  satisfying the conclusion, the global minimizer is still taken at one of the local families. Assume, by contradiction, that there is a sequence  $U_n$  approaching  $U$ , and for each  $H_n = K - U_n$ , there is some global minimizer  $\xi_n^{E_n}$  not from these families. By picking a subsequence, we can assume that it converges to a closed curve  $\xi_*$ , which belong one of the local families  $\xi_j^E$ . Using local uniqueness from Lemma A.13,  $\xi_n^{E_n}$  must belong to one of the local families as well. This is a contradiction.

To prove denseness, it suffices to prove that for a potential  $U$  satisfying the conclusion of Proposition A.14, we can make an arbitrarily small perturbation, such that there are no degenerate global minimizers.

Our strategy is to eliminate the degenerate global minimizers one by one using a sequence of perturbations. Let  $\tilde{\eta}_1^{E_1}, \dots, \tilde{\eta}_N^{E_N}$  be the set of all degenerate periodic orbits from Corollary A.2, and let  $\eta_1^{E_1}, \dots, \eta_N^{E_N}$  be their projection to  $\mathbb{T}^2$ . The set

of degenerate periodic orbits depends continuously on small perturbations to the Hamiltonian (Corollary A.2).

Let  $S \subset \{1, \dots, N\}$  denote the indices of the global  $(\rho_E, h)$ -minimizers among  $\eta_1^{E_1} \dots \eta_N^{E_N}$ . Note that for all  $j \notin S$ , either the homology of  $\eta_j^{E_j}$  is not  $h$ , or the homology of  $\eta_j^{E_j}$  is  $h$  but  $l_E(\eta_j^{E_j}) < l_E(h)$ . In either case, for sufficiently small perturbation to the potential  $U$ , we still have  $j \notin S$ .

Consider  $i \in S$ , we have  $l_{E_i}(\xi_i^{E_i}) = l_{E_i}(h)$ . We note that from Proposition A.14,  $\xi_i^{E_i}$  can never be the unique global minimizer. Indeed, since the local branch containing  $\xi_i^{E_i}$  cannot be continued to both sides of  $E_i$ , there is at least another local branch. Let  $V$  be a neighborhood of  $\xi_i^{E_i}$ , such that  $\bar{V}$  is disjoint from the set of other global minimizers with the same energy. For  $\delta > 0$  we define  $U_\delta : \mathbb{T}^2 \rightarrow \mathbb{R}$  such that  $U_\delta|_{\xi_i^{E_i}} = \delta$  and  $\text{supp } U_\delta \subset V$ . Let  $H_\delta = K - U - U_\delta$ , and let  $l_{E,\delta}$  be the perturbed length function. We have

$$l_{E_i,\delta}(\xi_i^{E_i}) = \int_{\xi_i^{E_i}} \sqrt{2(E_i + U + \delta)K} > l_{E_i}(\xi_i^{E_i}) = l_{E_i}(h) = l_{E_i,\delta}(h).$$

As a consequence,  $\xi_i^{E_i}$  is no longer a global minimizer for the perturbed system. Moreover, for sufficiently small  $\delta$ , no new degenerate global minimizer can be created. Hence the perturbation has decreased the number of degenerate global minimizers strictly. By repeating this process finitely many times can eliminate all degenerate global minimizers. □

### A.3 Proof of Theorem 4 about genericity of [DR1]-[DR3]

In this section we complete the plan laid out in the introduction to this section. We complete the proof of Theorem 26. This amounts to proving that finite local families of local minimizers, obtained from the previous section, are “in general position”.

We assume that the potential  $U_0 \in \mathcal{G}^r$  satisfies the conclusions of Theorem 26. Let  $\xi_j^{E,U}$  denote the branches of local minimizers from Theorem 26, where we have made the dependence on  $U$  explicit. There exists a neighbourhood  $V(U_0)$  of  $U_0$ , such that the local branches  $\xi_j^{E,U}$  are defined for  $E \in [a_j - \sigma/2, b_j + \sigma/2]$  and  $U \in V(U_0)$ .

Define a set of functions

$$f_j : [a_j - \sigma/2, b_j + \sigma/2] \times V(U_0) \rightarrow \mathbb{R}, \quad f_j(E, U) = l_E(\xi_j^{E,U}).$$

Then  $\xi_i^{E,U}$  is a global minimizer if and only if

$$f_i(E, U) = f_{\min}(E, U) := \min_j f_j(E, U),$$



where the minimum is taken over all  $j$ 's such that  $E \in [a_j, b_j]$ .

The following proposition implies Theorem 4.

**Proposition A.15.** *There exists an open and dense subset  $V'$  of  $V(U_0)$  such that for every  $U \in V'$ , the following hold:*

1. *For each  $E \in [E_0, \bar{E}]$ , there at at most two  $j$ 's such that  $f_j(E, U) = f_{\min}(E, U)$ ;*
2. *There are at most finitely many  $E \in [E_0, \bar{E}]$  such that there are two  $j$ 's with  $f_j(E, U) = f_{\min}(E, U)$ ;*
3. *For any  $E \in [E_0, \bar{E}]$  and  $j_1, j_2$  be such that  $f_{j_1}(E, U) = f_{j_2}(E, U) = f_{\min}(E, U)$ ; we have*

$$\frac{\partial}{\partial E} f_{j_1}(E, U) \neq \frac{\partial}{\partial E} f_{j_2}(E, U).$$

*Proof.* We first note that it suffices to prove the theorem under the additional assumption that all functions  $f_j$ 's are defined on the same interval  $(a, b)$  with  $f_{\min}(E, U) = \min_j f_j(E, U)$ . Indeed, we may partition  $[E_0, \bar{E}]$  into finitely many intervals, on which the number of local branches is constant, and prove proposition on each interval.

We define a map

$$f = (f_1, \dots, f_N) : (a, b) \times V(U_0) \longrightarrow \mathbb{R}^N,$$

and subsets

$$\Delta_{i_1, i_2, i_3} = \{(x_1, \dots, x_n); x_{i_1} = x_{i_2} = x_{i_3}\}, \quad 1 \leq i_1 < i_2 < i_3 \leq N,$$

$$\Delta_{i_1, i_2} = \{(x_1, \dots, x_n); x_{i_1} = x_{i_2}\}, \quad 1 \leq i_1 < i_2 \leq N$$

of  $\mathbb{R}^N \times \mathbb{R}^N$ . We also write  $f^U(E) = f(E, U)$ . The following two claims imply our proposition:

1. *For an open and dense set of  $U \in V(U_0)$ , for all  $1 \leq i_1 < i_2 < i_3 \leq N$ , the set  $(f^U)^{-1} \Delta_{i_1, i_2, i_3}$  is empty.*
2. *For an open and dense set of  $U \in V(U_0)$ , and all  $1 \leq i_1 < i_2 \leq N$ , the map  $f^U : (a, b) \longrightarrow \mathbb{R}^N$  is transversal to the submanifold  $\Delta_{i_1, i_2}$ .*

Indeed, the first claim imply the first statement of our proposition. It follows from our second claim that there are at most finitely many points in  $(f^U)^{-1} \Delta_{i_1, i_2}$ , which implies the second statement. Furthermore, using the second claim, we have for any  $E \in (f^U)^{-1} \Delta_{i_1, i_2}$ , the subspace  $(Df^U(E))\mathbb{R}$  is transversal to  $T\Delta_{i_1, i_2}$ . This implies the third statement of our proposition.

For a fixed energy  $E$  and  $(v_1, \dots, v_N) \in \mathbb{R}^N$ , let  $\delta U : \mathbb{T}^2 \rightarrow \mathbb{R}$  be such that  $\delta U(\varphi) = v_j$  on an open neighbourhood of  $\xi_j^E$  for each  $j = 1, \dots, N$ . Let  $l_{E,\epsilon}$  and  $\xi_j^{E,\epsilon}$  denote the arclength and local minimizer corresponding to the potential  $U + \epsilon \delta U$ . For any  $\varphi$  in a neighbourhood of  $\xi_j^E$ , we have

$$E + U(\varphi) + \delta U(\varphi) = E + U(\varphi) + \epsilon v_j,$$

hence for sufficiently small  $\epsilon > 0$ ,  $\xi_j^{E,\epsilon} = \xi_j^{E+\epsilon v_j}$ .

The directional derivative

$$D_U f(E, U) \cdot \delta U = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} l_{E,\epsilon}(\xi_j^{E,\epsilon}) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} l_{E+\epsilon v_j}(\xi_j^{E+\epsilon v_j}) = \frac{\partial}{\partial E} f_j(E, U) v_j.$$

It follows from a direct computation that each  $f_j$  is strictly increasing in  $E$  and the derivative in  $E$  never vanishes. As a consequence, we can choose  $(v_1, \dots, v_N)$  in such a way, that  $D_U f(E, U) \cdot \delta U$  takes any given vector in  $\mathbb{R}^N$ . This implies the map

$$D_U f : (a, b) \times TV(U_0) \rightarrow \mathbb{R}^N$$

has full rank at any  $(E, U)$ . As a consequence,  $f$  is transversal to any  $\Delta_{i_1, i_2, i_3}$  and  $\Delta_{i_1, i_2}$ . Using Sard's lemma, we obtain that for a generic  $U$ , the image of  $f^U$  is disjoint from  $\Delta_{i_1, i_2, i_3}$  and that  $f^U$  is transversal to  $\Delta_{i_1, i_2}$ .  $\square$

## A.4 Proof of Theorem 5

Now we prove the second Theorem about genericity of properties of geodesic flows. Proving Theorem 5 consists of two steps consisting of two localized perturbations of the potential  $U$ .

First, we perturb  $U$  near the origin to achieve properties [A1]–[A2]. Then we perturb it away from the origin around a point on a homoclinic orbit  $\gamma^+$  and satisfy [A3]–[A4].

Let  $W'$  be a  $\rho$ -neighborhood of the origin in  $\mathbb{R}^2$  for small enough  $\rho > 0$  so that it does not intersect sections  $\Sigma_\pm^s$  and  $\Sigma_\pm^u$ . Consider  $\xi(\theta)$  a  $C^\infty$ -bump function so that  $\xi(\theta) \equiv 1$  for  $|\theta| < \rho/2$  and  $\xi(\theta) \equiv 0$  for  $|\theta| > \rho$ . Let  $Q_\zeta(\theta) = \sum \zeta_{ij} \theta_i \theta_j$  be a symmetric quadratic form. Consider  $U_\zeta(\theta) = U(\theta) + \xi(\theta)(Q_\zeta(\theta) + \zeta_0)$ . In  $W' \times \mathbb{R}^2$  we can diagonalize both: the quadratic form  $K(p) = \langle Ap, p \rangle$  and the Hessian  $\partial^2 U(0)$ . Explicit calculation shows that choosing properly  $\zeta$  one can make the minimum of  $U$  at 0 being unique and eigenvalues to be distinct.

Suppose now that conditions [A1]–[A2] hold. We perturb  $U$  and satisfy [A3]–[A4]. Fix a point  $\theta^* \in \gamma^+$  at a distance of order of one from the origin. In particular, it is away from sections  $\Sigma_\pm^s$ . Let  $W''$  be its small neighborhood so that intersects only one homoclinic  $\gamma^+$ . Denote  $w^u = W^u \cap \Sigma_+^u$  an unstable curve on the exit section  $\Sigma_+^u$

and  $w^s = W^s \cap \Sigma_+^s$  a stable curve on the enter section  $\Sigma_+^s$ . Denote on  $w^u$  (resp.  $w^s$ ) the point of intersection  $\Sigma_+^u$  (resp.  $\Sigma_+^s$ ) with strong stable direction  $q^{ss}$  (resp.  $q^{uu}$ ). Recall that  $q^+$  (resp.  $p^+$ ) denotes the point of intersection of  $\gamma^+$  with  $\Sigma_+^u$  (resp.  $\Sigma_+^s$ ). We also denote by  $T^{uu}(q^+)$  (resp.  $T^{ss}(p^+)$ ) subspaces the tangent to  $w^u$  (resp.  $w^s$ ) at the corresponding points. The critical energy surface  $\{H = K - U = 0\}$  is denoted by  $S_0$ . In order to satisfy condition [A3]–[A4] the global map  $\Phi_{glob}^+$  has satisfy

- $\Phi_{glob}^+ w^u \cap w^s \neq q^{ss}$  and  $(\Phi_{glob}^+)^{-1}(w^s) \cap w^u \neq q^{uu}$ .
- $D\Phi_{glob}^+(q^+)|_{TS_0} T^{uu}(q^+) \pitchfork T^{ss}(p^+)$ ,  $D\Phi_{glob}^-(q^-)|_{TS_0} T^{uu}(q^-) \pitchfork T^{ss}(p^-)$ .

The first condition can also be viewed as a property of the restriction of  $\Phi_{glob}^+|_{S_0}$ . Notice that  $\Phi_{glob}^+$ , restricted to  $S_0$ , is a 2-dimensional map.

Consider perturbations  $\delta U \in \mathcal{G}^r$  of the potential  $U$  localized in  $W''$ . By Lemma A.4 the differential map  $D_U \Phi$  generates a subspace orthogonal to the gradient  $\nabla H^U$  and the Hamiltonian vector field  $\chi^U(\cdot)$  of  $H^U$ . Notice that when we restrict  $\Phi_{glob}^+$  onto  $\Sigma_+^u \cap S_0$  we factor out  $\nabla H^U$  and  $\chi^U(\cdot)$ . Both conditions on  $\Phi_{glob}^+$  (resp.  $D\Phi_{glob}^+|_{S_0}$ ) are non-equality conditions on images and preimages for a 2-dimensional map. Thus, these conditions can be satisfies by Lemma A.4.

## B Derivation of the slow mechanical system

In this section we consider the system

$$H_\epsilon(\theta, p, t) = H_0(p) + \epsilon H_1(\theta, p, t)$$

near a double resonance  $p_0 = \Gamma_{\vec{k}} \cap \Gamma_{\vec{k}'}$ . Note that this implies

$$(\vec{k}_1, k_0) \cdot (\partial_p H_0(p_0), 1) = (\vec{k}'_1, k'_0) \cdot (\partial_p H_0(p_0), 1) = 0,$$

where  $\vec{k} = (\vec{k}_1, k_0)$  and  $\vec{k}' = (\vec{k}'_1, k'_0)$ . In particular, we have  $\vec{k}_1 \nparallel \vec{k}'_1$ . We may choose  $\vec{k}'$  differently without changing the double resonance  $p_0$ , such that

$$\det B = 1, \quad B = \begin{bmatrix} \vec{k}_1 \\ \vec{k}'_1 \end{bmatrix}, \quad (51)$$

with  $\vec{k}_1, \vec{k}'_1$  viewed as row vectors.

We will describe a series of coordinate changes and rescalings that reduce the system to a perturbation of the slow system.

In section B.1, we describe a resonant normal form.

In section B.2, we describe the affine coordinate change and the rescaling, revealing the slow system.

In section B.3, we discuss variational properties of these coordinate changes.

### B.1 Normal forms near double resonances

Write  $\omega_0 := \partial_p H_0(p_0)$ , then the orbit  $(\omega_0, 1)t$  is periodic. Let

$$T = \inf_{t>0} \{t(\omega_0, 1) \in \mathbb{Z}^3\}$$

be the minimal period, then there exists some constant  $c > 0$  such that  $T \leq c \|\vec{k}\| \|\vec{k}'\|$ .

Given a function  $f : \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T} \rightarrow \mathbb{R}$ , we define

$$[f]_{\omega_0}(\theta, p, t) = \frac{1}{T} \int_0^T f(\theta + \omega_0 s, p, t + s) ds.$$

$[f]$  corresponds to the resonant component related to the double resonance.

Writing  $H_1(\theta, p, t) = \sum_{\vec{l} \in \mathbb{Z}^3} h_{\vec{l}}(p) e^{2\pi i \vec{l} \cdot (\theta, t)}$ , and let  $\Lambda = \text{span}_{\mathbb{Z}}\{\vec{k}, \vec{k}'\} \subset \mathbb{Z}^3$ , we define

$$Z(\theta, p, t) = [H_1]_{\omega_0} = \sum_{\vec{l} \in \Lambda} h_{\vec{l}}(p) e^{2\pi i \vec{l} \cdot (\theta, t)}.$$

$Z$  only depends on  $\vec{k} \cdot (\theta, t)$ ,  $\vec{k}' \cdot (\theta, t)$ , and  $p$ .

We define a rescaled differential in the action variable by

$$\partial_I f(\theta, p, t) = \sqrt{\epsilon} \partial_p f(\theta, p, t),$$

and use the notation  $C_I^r$  to denote the  $C^r$  norm with  $\partial_p$  replaced by  $\partial_I$ .

**Theorem 27.** *Assume that  $r \geq 4$ . Then there exists a  $C^2$  coordinate change*

$$\Phi_\epsilon : \mathbb{T}^2 \times U_{E\sqrt{\epsilon}}(p_0) \times \mathbb{T} \longrightarrow \mathbb{T}^2 \times U_{2E\sqrt{\epsilon}}(p_0) \times \mathbb{T},$$

*which is the identity in the  $t$  component, and a constant*

*$\tilde{C} = \tilde{C}(\vec{k}, \vec{k}', \bar{E}, \|H_1\|_{C^r}, \|H_0\|_{C^r})$ , such that*

$$N_\epsilon(\theta, p, t) := H_\epsilon \circ \Phi_\epsilon(\theta, p, t) =$$

$$H_0(p) + \epsilon Z(\theta, p) + \epsilon Z_1(\theta, p) + \epsilon R(\theta, p, t),$$

*where  $Z_1 = [Z_1]_{\omega_0}$  and*

$$\|Z_1\|_{C_I^2} \leq \tilde{C}\sqrt{\epsilon}, \quad \|R\|_{C_I^2} \leq \tilde{C}\epsilon,$$

*and*

$$\|\Phi_\epsilon - Id\|_{C_I^2} \leq \tilde{C}\epsilon.$$

**Remark B.1.** *Our normal form is the cut-off from a formal series obtained by a sequence of “partial averaging”, see, for example expansion (6.5) in [8, Section 6.1.2]. While this expansion is classical, our goal here is to obtain precise control of the norms with minimal regularity assumptions. In particular, the norm estimate of  $\Phi_\epsilon - Id$  is stronger than the usual results, and is needed in the proof of Proposition B.6.*

The rest of this section is dedicated to proving Theorem 27. Denote  $\Pi_\theta(\theta, p, t) = \theta$ ,  $\Pi_p(\theta, p, t) = p$  the natural projections.

**Lemma B.1.** *We have the following properties of the rescaled norm.*

1.  $\|f\|_{C_I^r} \leq \|f\|_{C^r}$ ,  $\|f\|_{C^r} \leq \epsilon^{-r/2} \|f\|_{C_I^r}$ .
2.  $\|\partial_\theta f\|_{C_I^{r-1}} \leq \|f\|_{C_I^r}$ ,  $\|\partial_p f\|_{C_I^r} \leq \frac{1}{\sqrt{\epsilon}} \|f\|_{C_I^r}$ .
3.  $\|fg\|_{C_I^r} \leq \|f\|_{C_I^r} \|g\|_{C_I^r}$ .
4. Let  $\Phi = (\Phi_\theta, \Phi_p, Id) : \mathbb{T}^2 \times U \times \mathbb{T} \longrightarrow \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T}$  be a smooth mapping. Then

$$\|f \circ \Phi\|_{C_I^r} \leq \|f\|_{C_I^r} \left( \max\{\|\Pi_\theta \Phi\|_{C_I^r}, \|\Pi_p \Phi\|_{C_I^r} / \sqrt{\epsilon}\} \right)^r.$$

5. Let  $\Phi$  be as above, then

$$\|f \circ \Phi\|_{C_I^r} \leq \|f \circ \Phi\|_{C^0} + \|f\|_{C_I^r} + \|f \circ (\Phi - Id)\|_{C_I^r}.$$

*Proof.* The first two conclusions follow directly from definition. For the third conclusion, we have  $\|\tilde{f}\|_{C^r} = \|f\|_{C_I^r}$ , where

$$\tilde{f}(\theta, I) = f(\theta, \sqrt{\epsilon}I).$$

The third conclusion follows from properties of the  $C^r$ -norm.

For the fourth conclusion, we note

$$f \circ \Phi = \tilde{f} \circ \tilde{\Phi},$$

where  $\tilde{f}$  is as before, and  $\tilde{\Phi}(\theta, I) = (\Phi_\theta(\theta, \sqrt{\epsilon}I), \Phi_p(\theta, \sqrt{\epsilon}I)/\sqrt{\epsilon})$ . We have  $\|\tilde{\Phi}\|_{C^r} = \max\{\|\Pi_\theta\Phi\|_{C_I^r}, \|\Pi_p\Phi\|_{C_I^r}/\sqrt{\epsilon}\}$ . The fourth conclusion follows a property of  $C^r$  functions known as the Faa-di Bruno formula  $\|f \circ \Phi\|_{C^r} \leq c\|f\|_{C^r}\|\Phi\|_{C^r}^r$  for some  $c = c(r)$ , see e.g. [29].

For the fifth conclusion, since the differential operator is linear,

$$\begin{aligned} \|f \circ \Phi\|_{C_I^r} &\leq \|f \circ \Phi\|_{C^0} + \|\partial_{(\theta, I)}(f \circ \Phi)\|_{C_I^{r-1}} \\ &\leq \|f \circ \Phi\|_{C^0} + \|\partial_{(\theta, I)}f\|_{C_I^{r-1}} + \|\partial_{(\theta, I)}(f \circ (\Phi - Id))\|_{C_I^{r-1}}. \end{aligned}$$

The estimate follows.  $\square$

We reserve the notations  $c$  for a unspecified absolute constant, and  $\tilde{C}$  for a unspecified constant depending on  $\vec{k}, \vec{k}', \bar{E}, \|H_1\|_{C^r}, \|H_0\|_{C^r}$ . For  $\rho > 0$ , denote

$$D_\rho = \mathbb{T}^2 \times U_\rho(p_0) \times \mathbb{T} \times \mathbb{R}.$$

Our main technical tool is an adaptation of an inductive lemma due to Bounemoura.

**Lemma B.2.** *Assume  $r \geq 4$ ,  $\rho > 0$ ,  $\mu > 0$  satisfies*

$$0 < \epsilon \leq \mu^2, \quad T\mu < 1, \quad (T\mu)\sqrt{\epsilon} \leq c\frac{\rho}{2(r-2)}.$$

*Assume that*

$$H : \mathbb{T}^2 \times U_\rho(p_0) \times \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad H(\theta, p, t, E) = l + g_0 + f_0,$$

*where  $l(p, E) = (\omega_0, 1) \cdot (p, E)$  is linear,  $g_0, f_0$  are  $C^r$  and depend only on  $(\theta, p, t)$ , and*

$$\|\partial_\theta g_0\|_{C_I^{r-1}(D_\rho)} \leq c\sqrt{\epsilon}\mu, \quad \|\partial_p g_0\|_{C_I^{r-1}(D_\rho)} \leq c\mu,$$

$$\|f_0\|_{C_I^r(\rho_0)} \leq c\sqrt{\epsilon}\mu, \quad \|\partial_p f_0\|_{C_I^{r-1}} \leq c\sqrt{\epsilon}\mu.$$

Then for  $j \in \{0, \dots, r-2\}$  and  $\rho_j = \rho - j\frac{\rho}{2(r-2)} > \rho/2$ , there exists a collection of  $C^2$ -symplectic maps  $\Phi_j : D_{\rho_j} \longrightarrow D_\rho$ , of the special form

$$\Phi_j(\theta, p, t, E) = (\Theta(\theta, p, t), P(\theta, p, t), t, E + \tilde{E}(\theta, p, t)).$$

The maps  $\Phi_j$  have the properties

$$\|\Pi_\theta(\Phi_j - Id)\|_{C_I^2(D_{\rho_j})} \leq c(T\mu)^2, \quad \|\Pi_p(\Phi_j - Id)\|_{C_I^2(D_{\rho_j})} \leq c(T\mu)\sqrt{\epsilon},$$

and

$$H \circ \Phi_j = l + g_j + f_j,$$

for each  $j \in \{0, \dots, r-3\}$  satisfying  $g_{j+1} = g_j + [f_j]_{\omega_0}$  and

$$\|\partial_\theta g_j\|_{C^{r-j-1}(D_{\rho_j})} \leq c\sqrt{\epsilon}\mu, \quad \|\partial_p g_j\|_{C^{r-j-1}(D_{\rho_j})} \leq c\mu,$$

$$\|f_j\|_{C^{r-j}(D_{\rho_j})} \leq c(T\mu)^j \sqrt{\epsilon}\mu.$$

*Proof.* The proof is an adaptation of the proof of Proposition 3.2, [18], page 9.

Following [18], we define

$$\chi_j = \frac{1}{T} \int_0^T t(f_j - [f_j]_{\omega_0})(\theta + \omega_0 s, p, t + s) ds$$

and

$$\Phi_{j+1} = \Phi_j \circ \Phi_1^{\chi_j},$$

where  $\Phi_s^{\chi_j}$  is the time- $s$  map of the Hamiltonian flow of  $\chi_j$ .

Using the fact that  $\chi_j$  is independent of  $E$ , we have the map  $\Phi^{\chi_j}$  is independent of  $E$  in the  $(\theta, p, t)$  components. Furthermore,  $\Phi^{\chi_j}$  is the identity in the  $t$  component, and  $\Pi_E \Phi^{\chi_j} - E$  is independent of  $E$ . Hence  $\Phi_j$  takes the special format described in the lemma. The special form of  $\Phi_j$  implies that  $g_j$  and  $f_j$  are also independent of  $E$ , allowing the induction to continue.

We now make several norm estimates.

$$\|[f_j]\|_{C_I^r} \leq \|f_j\|_{C_I^r} \leq c(T\mu)^j \sqrt{\epsilon}\mu, \quad \|\chi_j\|_{C_I^r} \leq T\|f_j\|_{C_I^r} \leq c(T\mu)^{j+1} \sqrt{\epsilon}.$$

For  $j \geq 1$ , using the inductive assumption,

$$\|\partial_p \chi_j\|_{C_I^{r-j-1}} \leq \frac{1}{\sqrt{\epsilon}} \|\chi_j\|_{C_I^{r-j}} \leq c(T\mu)^{j+1} c(T\mu)^2,$$

while for  $j = 0$ , the initial assumption on  $f_0$  implies

$$\|\partial_p \chi_0\|_{C_I^{r-1}} \leq T \|\partial_p f\|_{C_I^{r-1}} \leq cT\sqrt{\epsilon}\mu \leq c(T\mu)$$

As a consequence

$$\|\Pi_\theta(\Phi^{\chi_j} - Id)\|_{C_I^{r-j-1}} \leq c \|\partial_p \chi_j\|_{C_I^{r-j-1}} \leq c(T\mu)^2,$$

$$\|\Pi_p(\Phi^{\chi_j} - Id)\|_{C_I^{r-j-1}} \leq c \|\partial_\theta \chi_j\|_{C_I^{r-j-1}} \leq c(T\mu)^{j+1}\sqrt{\epsilon} \leq c(T\mu)\sqrt{\epsilon}.$$

The assumption  $(T\mu)\sqrt{\epsilon} \leq c\rho/(2r-4)$  implies that  $\{\Phi^{\chi_j}\}_j$  are well define maps from  $D_{\rho_{j+1}}$  to  $D_{\rho_j}$ . The norm estimate for  $\Phi_j$  follows from that of  $\Phi^{\chi_j}$ .

We define  $g_{j+1} = g_j + [f_j]_{\omega_0}$ . The norm estimate for  $\|g_{j+1}\|$  follows directly from the inductive assumption on  $\|g_j\|$  and  $\|f_j\|$ . By a standard computation, we have

$$f_{j+1} = \int_0^1 \{f_j^s, \chi_j\} \circ \Phi_s^{\chi_j} ds,$$

where  $f_j^s = g_j + sf_j + (1-s)[f_j]_{\omega_0}$ .

It's easy to see that  $g_j$  is the dominant term in  $f_j^s$  and all estimates of  $g_j$  carry over to  $f_j^s$  with possibly different constants. We have

$$\begin{aligned} \|\{f_j^s, \chi_j\}\|_{C_I^{r-j-1}} &\leq \|\partial_\theta f_j^s\|_{C_I^{r-j-1}} \|\partial \chi_j\|_{C_I^{r-j-1}} + \|\partial_p f_j^s\|_{C_I^{r-j-1}} \|\partial_\theta \chi_j\|_{C_I^{r-j-1}} \\ &\leq c\sqrt{\epsilon}\mu \frac{1}{\sqrt{\epsilon}} \|\chi\|_{C_I^{r-j}} + c\mu \|\chi\|_{C_I^{r-j}} \leq c(T\mu)^{j+1}\sqrt{\epsilon}\mu. \end{aligned}$$

Furthermore, by Lemma B.1, items 4 and 5,

$$\begin{aligned} &\|\{f_j^s, \chi_j\} \circ \Phi_s^{\chi_j}\|_{C_I^{r-j-1}} \\ &\leq \|\{f_j^s, \chi_j\} \circ \Phi_s^{\chi_j}\|_{C^0} + \|\{f_j^s, \chi_j\}\|_{C_I^{r-j-1}} + \|\{f_j^s, \chi_j\} \circ (\Phi^{\chi_j} - Id)\|_{C_I^{r-j-1}} \\ &\leq \|\{f_j^s, \chi_j\}\|_{C_I^{r-j-1}} + \|\{f_j^s, \chi_j\}\|_{C_I^{r-j-1}} \left( \max\{cT^2\mu^2, c(T\mu)\} \right)^{r-j-1} \\ &\leq c \|\{f_j^s, \chi_j\}\|_{C_I^{r-j-1}} \leq c(T\mu)^{j+1}\sqrt{\epsilon}\mu. \end{aligned}$$

The norm estimate for  $f_{j+1}$  follows, and the induction is complete.  $\square$

*Proof of Theorem 27.* We write

$$H(\theta, p, t, E) = H_\epsilon(\theta, p, t) - H_0(p_0) + E = l + g_0 + f_0,$$

where  $l(p, E) = (\omega_0, 1) \cdot (p, E)$ ,

$$g_0(\theta, p, t) = H_0(p) - H_0(p_0) - \omega_0 \cdot p$$



and  $f_0 = \epsilon H_1$ .

Define  $\rho = 2\bar{E}\sqrt{\epsilon}$  and  $\mu = \bar{E}\sqrt{E}$ . We have

$$\|\partial_\theta g_0\|_{C_I^{r-1}(D_\rho)} = 0 \leq c\sqrt{\epsilon}\mu, \quad \|\partial_p g_0\|_{C_I^{r-1}(D_\rho)} \leq c\bar{E}\sqrt{\epsilon} = c\mu,$$

$$\|f_0\|_{C_I^r(D_\rho)} \leq \epsilon \leq c\sqrt{\epsilon}\mu, \quad \|\partial_p f\|_{C_I^{r-1}(D_\rho)} \|\partial_p f\|_{C^r(D_\rho)} \leq c\sqrt{\epsilon}\mu.$$

By choosing  $\epsilon$  sufficiently small depending on  $T$ , we have  $(T\mu)\sqrt{\epsilon} = T\bar{E}^2\epsilon \leq c\rho/(2r-4)$  and  $T\mu < 1$ . The conditions of Lemma B.2 are satisfied, and we apply the lemma with  $r = 4$ . Since with our choice of parameters  $(T\mu)\sqrt{\epsilon} < (T\mu)^2$ , there exists a map  $\Phi_2 : D_{\rho/2} \rightarrow D_\rho$ ,

$$\|\Phi_2 - Id\|_{C_I^2(D_{\rho/2})} \leq c(T\mu)^2 \leq cT^2\bar{E}^2\epsilon,$$

and

$$H \circ \Phi_2 = l + g_0 + [f_0]_{\omega_0} + [f_1]_{\omega_0} + f_2,$$

with

$$\|[f_1]_{\omega_0}\|_{C_I^2(D_{\rho/2})} \leq c(T\mu)\sqrt{\epsilon}\mu = cT\bar{E}^2\epsilon^{\frac{3}{2}},$$

$$\|f_2\|_{C^2(D_{\rho/2})} \leq c(T\mu)^2\sqrt{\epsilon}\mu = cT^2\bar{E}^3\epsilon^2.$$

Using  $l + g_0 = H_0(p) + \epsilon Z - H_0(p_0)$ , define  $\epsilon Z = [f_0]_{\omega_0}$ ,  $\epsilon Z_1 = [f_1]_{\omega_0}$ , and  $\epsilon R = f_2$ , we obtain

$$(H_\epsilon + E) \circ \Phi_2 = H_0 + \epsilon Z + \epsilon Z_1 + \epsilon R + E$$

with the desired estimates. Finally, we define  $\Phi_\epsilon(\theta, p, t) = \Phi_2(\theta, p, t, E)$ . This is well defined since  $\Phi_2$  is independent of  $E$ .  $\square$

## B.2 Affine coordinate change and rescaling

We first make a coordinate change  $\bar{p} = p - p_0$ , shifting the double resonance to  $\bar{p} = 0$ . Formally,

$$S_{p_0}(\theta, \bar{p}, t, E) = (\theta, \bar{p} + p_0, t, E).$$

We then make a linear change of coordinate by writing  $\varphi^s = (\vec{k}_1 \cdot \theta + k_0 t, \vec{k}'_1 \cdot \theta + k'_0 t)$ , and then complete it to a symplectic coordinate change. Using the matrix  $B$ , defined in (51), and symplecticity, formally, we have

$$\begin{bmatrix} \theta \\ t \\ \bar{p} \\ E \end{bmatrix} = L \begin{bmatrix} \varphi^s \\ t \\ p^s \\ E' \end{bmatrix} = \begin{bmatrix} B^{-1} & -B^{-1} \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix} \\ 0 & 1 \\ & & B^T & 0 \\ & & k_0, k'_0 & 1 \end{bmatrix} \begin{bmatrix} \varphi^s \\ t \\ p^s \\ E' \end{bmatrix}.$$

Denote  $\Phi_L = S_{p_0} \circ L$ . Notice that this formula implicitly defines  $p^s$ .

**Lemma B.3.** *In the notations of Theorem 27 for the Hamiltonian  $N_\epsilon = H_\epsilon \circ \Phi_\epsilon$  in the normal form we have*

$$(N_\epsilon + E) \circ \Phi_L - E' = H_0(p_0) + K(p^s) - \epsilon U(\varphi^s) + \epsilon P_0(\varphi^s, p^s) + \epsilon R(\varphi^s, p^s, t),$$

where

$$K(p^s) = \langle \partial_{pp}^2 H_0(p_0) B^T p^s, B^T p^s \rangle, \quad U(\varphi^s) = -Z(\varphi^s, p_0), \quad (52)$$

and  $\|P_0\|_{C_I^2} \leq \tilde{C}\sqrt{\epsilon}$ ,  $\|R\|_{C_I^2} \leq \tilde{C}\epsilon$ .

*Proof.* Notice that by definition of the above matrix  $B$  we have  $\bar{p} = p - p_0 = B^T p^s$ . Consider  $(N_\epsilon + E) \circ \Phi_L$  and expand  $H_0(p) = H_0(p_0 + \bar{p}) = H_0(p_0 + B^T p^s)$  and  $Z(\varphi^s, p_0 + \bar{p})$  near  $p_0$ . We have

$$\begin{aligned} & (N_\epsilon + E) \circ \Phi_L \\ &= H_0(p_0 + \bar{p}) + \epsilon Z(\varphi^s, p_0 + \bar{p}) + \epsilon Z_1(\varphi^s, p) + \epsilon R + (k_0, k'_0) \cdot p^s + E' \\ &= H_0(p_0) + \partial_p H_0(p_0) \bar{p} + \langle \partial_{pp}^2 H_0(p_0) \bar{p}, \bar{p} \rangle + \tilde{H}_0(\bar{p}) \\ &+ \epsilon \{Z(\varphi^s, p_0) + (Z(\varphi^s, p) - Z(\varphi^s, p_0))\} + \epsilon Z_1(\varphi^s, p) + \epsilon R + (k_0, k'_0) \cdot p^s + E' \\ &= \text{Combine } H_0, p\text{-terms} \quad H_0(p_0) + \partial_p H_0(p_0) \cdot B^T p^s + (k_0, k'_0) \cdot p^s + \\ &+ \langle \partial_{pp}^2 H_0(p_0) B^T p^s, B^T p^s \rangle + \tilde{H}_0(\bar{p}) \\ &+ \epsilon Z(\varphi^s, p_0) + \epsilon \tilde{Z}(\varphi^s, p) + \epsilon Z_1(\varphi^s, p) + \epsilon R + E' \\ &= H_0(p_0) + K(p^s) - \epsilon U(\varphi^s) + \epsilon P_0 + \epsilon R + E', \end{aligned}$$

where

$$\begin{aligned} \epsilon P_0(\varphi^s, p^s) &= \epsilon \tilde{Z}(\varphi^s, p) + \epsilon Z_1(\varphi^s, p) + \tilde{H}_0(\bar{p}), \\ \tilde{Z}(\varphi^s, p^s) &= Z(\varphi^s, p_0 + \bar{p}) - Z(\varphi^s, p_0), \end{aligned}$$

and

$$\tilde{H}_0(\bar{p}) = H_0(p_0 + \bar{p}) - H_0(p_0) - \partial_p H_0(p_0) \cdot \bar{p} - \langle \partial_{pp}^2 H_0(p_0) \bar{p}, \bar{p} \rangle$$

which is the third order Taylor remainder of  $H_0(p_0 + \bar{p})$ . In the third equality, we have used the fact that

$$0 = \dot{\varphi}^s = B\dot{\theta} + \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix} = B[\partial_p H_0(p_0)]^T + \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix} = 0,$$

whose transpose justifies the cancellation.

Finally, by direct calculation, we have

$$\|\tilde{H}_0\|_{C_I^2}, \|\epsilon \tilde{Z}(\varphi^s, p^s)\|_{C_I^2} \leq c\epsilon^{\frac{3}{2}},$$

and  $\|Z_1\|_{C_I^2} \leq \tilde{C}\sqrt{\epsilon}$ ,  $\|R\|_{C_I^2} \leq \tilde{C}\epsilon$  by Theorem 27.  $\square$

We write  $N_\epsilon^s(\varphi^s, p^s, t) = (N + E) \circ \Phi_L(\varphi^s, t, p^s, E') - E'$ .

To define the slow system precisely, we perform some rescalings to the system. Given a Hamiltonian  $H(\varphi^s, p^s, t)$ , we define

$$\bar{H}(\varphi^s, I^s, t) = \mathcal{S}_1(H)(\varphi^s, I^s, t) = \frac{1}{\sqrt{\epsilon}} H(\varphi^s, \sqrt{\epsilon} I^s, t),$$

$$\tilde{H}(\varphi^s, I^s, \tau) = \mathcal{S}_2(\bar{H})(\varphi^s, I^s, \tau) = \frac{1}{\sqrt{\epsilon}} \bar{H}(\varphi^s, I^s, \tau/\sqrt{\epsilon}).$$

Note that  $\tilde{H}$  is defined on  $\mathbb{T}^2 \times \mathbb{R}^2 \times \sqrt{\epsilon}\mathbb{T}$ . The flows of  $\bar{H}$  and  $\tilde{H}$  are both conjugate to that of  $H$ . Write  $\mathcal{S} = \mathcal{S}_2 \circ \mathcal{S}_1$ .

**Proposition B.4.** *We have*

$$H_\epsilon^s := \mathcal{S}(N_\epsilon^s) = H_0(p_0)/\epsilon + K(I^s) - U(\varphi^s) + \sqrt{\epsilon}P(\varphi^s, I^s, \tau), \quad (53)$$

where

$$\|P\|_{C^2(\varphi^s, I^s)} \leq \tilde{C}, \quad \|(\partial_{\varphi^s} P, \partial_{I^s} P)\|_{C^1(\varphi^s, I^s, \tau)} \leq \tilde{C}.$$

Here

$$\|P(\varphi^s, I^s, \tau)\|_{C^2(\varphi^s, I^s)} = \sup \|\partial_{\varphi^s}^2 P\|_{C^0}, \|\partial_{I^s}^2 P\|_{C^0}$$

is a  $C^2$  norm with  $\tau$  derivatives excluded, and  $C^1(\varphi^s, I^s, \tau)$  denote the normal  $C^1$  norm involving all variables.

*Proof.* We write

$$\sqrt{\epsilon}P(\varphi^s, I^s, \tau) = P_0(\varphi^s, \sqrt{\epsilon}I^s) + R(\varphi^s, \sqrt{\epsilon}I^s, \tau/\sqrt{\epsilon}).$$

We note that as differential operators,  $\partial_I = \partial_{I^s}$ , and

$$\partial_\tau R(\varphi^s, \sqrt{\epsilon}I^s, \tau/\sqrt{\epsilon}) = \sqrt{\epsilon}^{-1} \partial_t R(\varphi^s, \sqrt{\epsilon}I^s, \tau/\sqrt{\epsilon}).$$

The norm estimates follows from Lemma B.3. □

We denote

$$H^s(\varphi^s, I^s) = K(I^s) - U(\varphi^s), \quad (54)$$

this is the slow mechanical system. We will lift the functions  $H_\epsilon^s$  to  $\mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}$  without changing the name, and also regard  $H^s$  as a function on  $\mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}$  by adding trivial  $\tau$  dependence. With these notations, we have

**Proposition B.5.** *As  $\epsilon \rightarrow 0$ ,  $H_\epsilon^s - H_0(p_0)/\epsilon$  Tonelli converges to  $H^s$  as a uniform family (see section 10.2). Moreover,*

$$\partial_{(\varphi^s, I^s)} H_\epsilon^s \rightarrow \partial_{(\varphi^s, I^s)} H^s$$

in the  $C^1$  norm, and as a result, the Hamiltonian vector field of  $H_\epsilon^s$  converges to that of  $H^s$  in the  $C^1$  norm.

### B.3 Variational properties of the coordinate changes

We have made the following reductions from the original system  $H_\epsilon$  to the slow system  $H_\epsilon^s$ :  $N_\epsilon(\theta, p, t) = H_\epsilon \circ \Phi_\epsilon(\theta, p, t)$  is the normal form;  $N_\epsilon^s(\varphi^s, p^s, t) = N_\epsilon \circ \Phi_L(\varphi^s, p^s, t)$  incorporates an affine coordinate change;  $H_\epsilon^s(\varphi^s, I^s, \tau) = \mathcal{S}_2 \circ \mathcal{S}_1(N_\epsilon^s)(\varphi^s, I^s, \tau)$  is the result of two rescalings.

**Proposition B.6.** *We have the following relations between  $H_\epsilon$  and  $N_\epsilon$ :*

1.  $\alpha_{H_\epsilon}(c) = \alpha_{N_\epsilon}(c)$ ,  $\tilde{\mathcal{M}}_{H_\epsilon}(c) = \tilde{\mathcal{M}}_{N_\epsilon}(c)$ ,  $\tilde{\mathcal{A}}_{H_\epsilon}(c) = \tilde{\mathcal{A}}_{N_\epsilon}(c)$ ,  $\tilde{\mathcal{N}}_{H_\epsilon}(c) = \tilde{\mathcal{N}}_{N_\epsilon}(c)$ .
2.  $|A_{H_\epsilon, c}(\theta_1, \tilde{t}_1; \theta_2, \tilde{t}_2) - A_{N_\epsilon, c}(\theta_1, \tilde{t}_1; \theta_2, \tilde{t}_2)| \leq \tilde{C}\epsilon$ .
3.  $|h_{H_\epsilon, c}(\theta_1, t_1; \theta_2, t_2) - h_{N_\epsilon, c}(\theta_1, t_1; \theta_2, t_2)| \leq \tilde{C}\epsilon$ .

*Proof of Proposition B.6.* The symplectic invariance of the alpha function and the Mather, Aubry and Mañe sets follows from exactness.

Writing  $\Phi_\epsilon = (\Theta, P, t)$ , from Theorem 27, we have

$$\|\Phi_\epsilon - Id\|_{C^0} \leq \tilde{C}\epsilon, \quad \|\Phi_\epsilon - Id\|_{C^1} \leq \tilde{C}\sqrt{\epsilon}$$

By exactness of  $\Phi_\epsilon$ , we have there exists a function  $S : \mathbb{T}^2 \times U \times \mathbb{T} \rightarrow \mathbb{R}$  such that

$$Pd\Theta - pd\theta = dS.$$

We now estimate the  $C^0$ -norm of  $S$ . Write  $S_0 = P \cdot (\Theta - \theta)$ , this is a well defined smooth function on  $\mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T}$ . We compute

$$dS_0 = Pd\Theta - pd\theta + (p - P)d\theta + (\Theta - \theta)dP,$$

hence

$$dS = dS_0 + (p - P)d\theta + (\Theta - \theta)dP.$$

Since  $\|S_0\|_{C^0} \leq c\|\Phi_\epsilon - Id\|_{C^0} \leq \tilde{C}\epsilon$  and  $\|d(S - S_0)\|_{C^0} \leq \|\Phi_\epsilon - Id\|_{C^0}\|\Phi_\epsilon - Id\|_{C^1} \leq \tilde{C}\epsilon^{\frac{3}{2}}$ , we conclude that  $\|S\|_{C^0} \leq \tilde{C}\epsilon$ .

For the estimates of the action  $A_{N_\epsilon, c}(\theta_1, \tilde{t}_1; \theta_2, \tilde{t}_2)$ , let  $\gamma$  be its minimizer, and let  $\xi$  be a  $C^1$ -curve such that

$$\mathbb{L}^{-1}d\xi = \Phi_\epsilon(\mathbb{L}^{-1}d\gamma),$$

where, as before,  $d\gamma(t)$  denotes  $(\gamma(t), \dot{\gamma}(t), t)$ . Using exactness, we have

$$\begin{aligned} (L_{H_\epsilon} - c \cdot v)(d\xi) &= (pd\theta - H_\epsilon - cd\theta)(\mathbb{L}^{-1}d\xi) \\ &= (pd\theta - N_\epsilon - cd\theta + dS)(\mathbb{L}^{-1}d\gamma) = (L_{N_\epsilon} - c \cdot v)(\gamma) + dS(\mathbb{L}^{-1}d\gamma). \end{aligned}$$

Integrating, we have

$$|A_{N_\epsilon, c}(\theta_1, \tilde{t}_1; \theta_2, \tilde{t}_2) - A_{H_\epsilon, c}(\xi(\tilde{t}_1), \tilde{t}_1 \xi(\tilde{t}_2), \tilde{t}_2)| \leq C \|S\|_{C^0} \leq \tilde{C} \epsilon.$$

Since  $\|\Phi_\epsilon - Id\|_{C^0} \leq \tilde{C} \epsilon$ , we have  $\|\theta_1 - \xi(\tilde{t}_1)\|, \|\theta_2 - \xi(\tilde{t}_2)\| \leq \tilde{C} \epsilon$ . The estimate follows from the Lipschitz property of  $A_{H, c}$ .

Taking limit, we obtain the estimate for  $h_{H, c}$ .  $\square$

The relation between the normal form system  $N_\epsilon$  and the perturbed slow system  $H_\epsilon^s$  is summarized in Proposition 12.5. We split the proof of this proposition into two steps.

First, we have the following relations between  $N_\epsilon$  and  $N_\epsilon^s$ .

**Proposition B.7.** *Let  $c' = (B^T)^{-1}(c - p_0)$ .*

1.  $\alpha_{N_\epsilon}(c) = \alpha_{N_\epsilon^s}(c') - c' \cdot (k_0, k'_0)$ .
2.  $\tilde{\mathcal{M}}_{N_\epsilon}(c) = \Phi_L(\tilde{\mathcal{M}}_{N_\epsilon^s}(c')), \tilde{\mathcal{A}}_{N_\epsilon}(c) = \Phi_L(\tilde{\mathcal{A}}_{N_\epsilon^s}(c')), \tilde{\mathcal{N}}_{N_\epsilon}(c) = \Phi_L(\tilde{\mathcal{N}}_{N_\epsilon^s}(c'))$ .
3. Denote  $\varphi_i^s = B\theta_i + (k_0, k'_0)t \mod \mathbb{T}^2$ . We have

$$A_{N_\epsilon, c}(\theta_1, \tilde{t}_1; \theta_2, \tilde{t}_2) = A_{N_\epsilon^s, c'}(\varphi_1^s, \tilde{t}_1; \varphi_2^s, \tilde{t}_2).$$

4.  $h_{N_\epsilon, c}(\theta_1, t_1; \theta_2, t_2) = h_{N_\epsilon^s, c'}(\varphi_1^s, t_1; \varphi_2^s, t_2)$ .

*Proof.* The first two statements can be proved using symplectic invariance of these sets. Here, hence we provide an alternative proof based on the Lagrangian setting, which proves all four statements.

The angular components of the coordinate transform  $\Phi_L$  is given by

$$\Phi_L^1 \begin{bmatrix} \varphi^s \\ t \end{bmatrix} = \begin{bmatrix} B^{-1} & -B^{-1} \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi^s \\ t \end{bmatrix}, \quad (\Phi_L^1)^{-1} \begin{bmatrix} \theta \\ t \end{bmatrix} = \begin{bmatrix} B & \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ t \end{bmatrix}.$$

Given a curve  $\tilde{\gamma}^s : [\tilde{t}_1, \tilde{t}_2] \longrightarrow \mathbb{T}^2$ , define  $\gamma : [\tilde{t}_1, \tilde{t}_2] \longrightarrow \mathbb{T}^2$  by  $\gamma(t) = \Phi_L^a(\gamma^s)(t)$ . We have

$$\gamma^s(\tilde{t}_i) = \varphi_i^s \text{ iff } \gamma(\tilde{t}_i) = \theta_i, \quad i = 1, 2.$$

By definition, the one forms

$$(p - p_0)d\theta = (\Phi_L)^*(p^s d\varphi^s).$$

We have the following calculation:

$$\begin{aligned} L_{N_\epsilon}(d\gamma) &= (pd\theta - N_\epsilon)(\mathbb{L}^{-1}d\gamma) = ((p - p_0)d\theta - N_\epsilon dt)(\mathbb{L}^{-1}d\gamma) + p_0 \cdot \dot{\gamma} \\ &= (\Phi_L)^*(p^s d\varphi^s - N_\epsilon^s)(\mathbb{L}^{-1}d\gamma^s) + p_0 \cdot \dot{\gamma} = L_{N_\epsilon^s}(d\gamma^s) + p_0 \cdot \dot{\gamma}, \end{aligned}$$

where all expressions are evaluated at  $t \in (\tilde{t}_1, \tilde{t}_2)$ .

Using

$$\dot{\gamma} = B^{-1}\dot{\gamma}^s - B^{-1} \begin{bmatrix} k_0 \\ k'_0 \end{bmatrix},$$

we have

$$\begin{aligned} & (L_{N_\epsilon} - c \cdot v)(d\gamma) \\ &= L_{N_\epsilon}^s(d\gamma^s) - (c - p_0) \cdot \dot{\gamma} \\ &= L_{N_\epsilon}^s(d\gamma^s) - (c - p_0) \cdot (B^{-1}\dot{\gamma}^s + B^{-1} \cdot (k_0, k'_0)) \\ &= (L_{N_\epsilon}^s - c' \cdot v + c' \cdot (k_0, k'_0))(d\gamma^s). \end{aligned} \tag{55}$$

Note that by integrating along orbits, and using the ergodic theorem, (55) extends to invariant measures. Since

$$0 = \inf_{\mu} \int_{\mu} (L_{N_\epsilon} - c \cdot v + \alpha_{N_\epsilon}(c)) d\mu = \inf_{\mu} \int_{\mu} (L_{N_\epsilon}^s - c \cdot v + \alpha_{N_\epsilon}^s(c)) d\mu,$$

we obtain

$$\alpha_{N_\epsilon}(c) + c' \cdot (k_0, k'_0) = \alpha_{N_\epsilon}^s(c).$$

The relation between alpha functions, together with (55), implies

$$A_{N_\epsilon, c}(\theta_1, \tilde{t}_1; \theta_2, \tilde{t}_2) = A_{N_\epsilon, c'}(\varphi_1^s, \tilde{t}_1; \varphi_2^s, \tilde{t}_2).$$

By taking the limit, we obtain  $h_{N_\epsilon, c}(\theta_1, t_1; \theta_2, t_2) = h_{N_\epsilon, c'}(\varphi_1^s, t_1; \varphi_2^s, t_2)$ .

The third statement also implies that  $\gamma$  being semi-static (static) for  $(L_{N_\epsilon}, c)$  is equivalent to  $\gamma^s$  being semi-static (static) for  $(L_{N_\epsilon}^s, c')$ . Since

$$\Phi_L(\mathbb{L}^{-1}d\gamma^s) = \mathbb{L}^{-1}d\gamma,$$

the relation for the Aubry set and Mañe set follows.

The Mather set consists of the support of all invariant measures supported on the Aubry set (see e.g. [51]). The relation between Mather sets then follows from the relation between Aubry sets.  $\square$

Second, we have the following relations between  $N_\epsilon^s$  and  $H_\epsilon^s$ . Note that  $H_\epsilon^s$  is defined on  $\mathbb{T}^2 \times \sqrt{\epsilon} \mathbb{T}$ .

**Proposition B.8.** *Let  $\bar{c} = c'/\sqrt{\epsilon}$ .*

1.  $\alpha_{N_\epsilon^s}(c')/\epsilon = \alpha_{H_\epsilon^s}(\bar{c})$ .
2.  $\tilde{\mathcal{M}}_{N_\epsilon^s}(c') = \Phi_S(\tilde{\mathcal{M}}_{H_\epsilon^s}(\bar{c}))$ , etc., where  $\Phi_S(\varphi^s, I^s) = (\varphi^s, \sqrt{\epsilon} I^s)$ .

3. Let  $\tilde{\tau}_i = \sqrt{\epsilon} \tilde{t}_i$ ,  $i = 1, 2$ . Then

$$A_{H_\epsilon^s, \bar{c}}(\varphi_1^s, \tilde{\tau}_1; \varphi_2^s, \tilde{\tau}_2) = A_{N_\epsilon^s, c'}(\varphi_1^s, \tilde{t}_1; \varphi_2^s, \tilde{t}_2)/\sqrt{\epsilon}.$$

4. Let  $\tau_i = \sqrt{\epsilon} t_i$ ,  $i = 1, 2$ . Then

$$h_{H_\epsilon^s, \bar{c}}(\varphi_1^s, \tau_1; \varphi_2^s, \tau_2) = h_{N_\epsilon^s, c'}(\varphi_1^s, t_1; \varphi_2^s, t_2)/\sqrt{\epsilon}.$$

*Proof.* Given a curve  $\gamma^s : [\tilde{t}_1, \tilde{t}_2] \longrightarrow \mathbb{T}^2$ , define  $\bar{\gamma} : [\sqrt{\epsilon} \tilde{t}_1, \sqrt{\epsilon} \tilde{t}_2] \longrightarrow \mathbb{T}^2$  by  $\bar{\gamma}(\tau) = \gamma^s(\tau/\sqrt{\epsilon})$ . It follows that

$$\bar{\gamma}'(\tau) = \frac{1}{\sqrt{\epsilon}} \dot{\gamma}^s \left( \frac{\tau}{\sqrt{\epsilon}} \right).$$

We have used  $\cdot$  to denote  $t$  derivative and  $'$  to denote  $\tau$  derivative.

Using the definition of the rescalings, we have

$$L_{\mathcal{S}_1 \circ H}(\varphi^s, v, t) = L_H(\varphi^s, v, t)/\sqrt{\epsilon}, \quad L_{\mathcal{S}_2 \circ H}(\varphi^s, v, \tau) = L_H(\varphi^s, \sqrt{\epsilon} v, \tau/\sqrt{\epsilon})/\sqrt{\epsilon}.$$

It follows that

$$L_{H_\epsilon^s}(\varphi^s, v, \tau) = L_{N_\epsilon^s}(\varphi^s, \sqrt{\epsilon} v, \tau/\sqrt{\epsilon})/\epsilon.$$

We have

$$\begin{aligned} L_{H_\epsilon^s}(\bar{\gamma}, \bar{\gamma}', \tau) - \bar{c} \cdot \bar{\gamma}' &= (L_{N_\epsilon^s}(\bar{\gamma}, \sqrt{\epsilon} \bar{\gamma}', \tau/\sqrt{\epsilon}) - \bar{c} \epsilon \cdot \bar{\gamma}')/\epsilon \\ &= (L_{N_\epsilon^s}(\gamma^s, \dot{\gamma}^s, t) - c' \cdot \dot{\gamma}^s)/\epsilon, \end{aligned}$$

where  $t = \tau/\sqrt{\epsilon}$ . Statement 1, 3 and 4 follows from the above expression, similar to the proof of Proposition B.7. Since

$$\Phi_S(\mathbb{L}^{-1} d\bar{\gamma}) = \mathbb{L}^{-1} d\gamma^s,$$

statement 2 follows, again similar to the proof of Proposition B.7.  $\square$

## C Variational aspects of the slow mechanical system

In this section we study the variational properties of the slow mechanical system

$$H^s(\varphi^s, I^s) = K(I^s) - U(\varphi^s),$$

with  $\min U = U(0) = 0$ .

The main goal of this section is to derive some properties of the “channel”  $\bigcup_{E>0} \mathcal{LF}_\beta(\lambda_h^E h)$ , and information about the Aubry sets for  $c \in \mathcal{LF}_\beta(\lambda_h^E h)$ . These statements Propositions 4.1, 4.2 and 4.3, and justify the pictures described in Figure 13 and Figure 14.

- In section C.1, we show that each  $\mathcal{LF}_\beta(\lambda_h^E h)$  is an segment parallel to  $h^\perp$ .
- In section C.2, we provide a characterization of the segment, and provide information about the Aubry sets.
- In section C.3, we provide a condition for the “width” of the channel to be non-zero.
- In section C.4, we discuss the limit of the set  $\mathcal{LF}_\beta(\lambda_h^E h)$  as  $E \rightarrow 0$  which corresponds to the “bottom” of the channel.

We drop all superscripts “s” to simplify the notations. The results proved in this section are closely related to discussions of Mather in [59]. However, it is difficult to locate exact references for the statements needed, so we include statements and proofs for completeness.

### C.1 Relation between the minimal geodesics and the Aubry sets

Assume that  $H(\varphi, I)$  satisfies the conditions [DR1]-[DR3] and the conditions [A0]-[A4]. Then for  $E \neq E_j$ ,  $1 \leq j \leq N-1$ , there exists a unique shortest geodesic  $\gamma_h^E$  for the metric  $g_E$  in the homology  $h$ . For the bifurcation values  $E = E_j$ , there are two shortest geodesics  $\gamma_h^E$  and  $\bar{\gamma}_h^E$ .

The function  $l_E(h)$  denotes the length of the shortest  $g_E$ -geodesic in homology  $h$ .  $l_E(h)$  is continuous and strictly increasing on  $E \geq 0$ , is positive homogeneous ( $l_E(nh) = nl_E(h)$ ,  $n \in \mathbb{N}$ ) and sub-additive ( $l_E(h_1 + h_2) \leq l_E(h_1) + l_E(h_2)$ ) in  $h$ .

Assume that the curves  $\gamma_h^E$  are parametrized using the Maupertuis principle. Let  $T(\gamma_h^E)$  be the period under this parametrization, and write  $\lambda(\gamma_h^E) = 1/(T(\gamma_h^E))$ .



We pick another vector  $\bar{h} \in H_1(\mathbb{T}^2, \mathbb{Z})$  such that  $h, \bar{h}$  form a basis of  $H_1(\mathbb{T}^2, \mathbb{Z})$  and for the dual basis  $h^*, \bar{h}^*$  in  $H^1(\mathbb{T}^2, \mathbb{R})$  we have  $\langle h, \bar{h}^* \rangle = 0$ . We denote  $\bar{h}^*$  by  $h_1^\perp$  to emphasise the latter fact.

**Theorem 28.** 1. For  $E = E_j$ ,

$$\mathcal{LF}_\beta(\lambda(\gamma_h^E) \cdot h) = \mathcal{LF}_\beta(\lambda(\bar{\gamma}_h^E) \cdot h).$$

As a consequence, write  $\lambda_h^E = \lambda(\gamma_h^E)$ , then the set  $\mathcal{LF}_\beta(\lambda_h^E h)$  is well defined (the definition is independent of the choice of  $\gamma_h^E$ ).

2. For each  $E > 0$ , there exists  $-\infty \leq a_E^-(h) \leq a_E^+(h) \leq \infty$  such that

$$\mathcal{LF}_\beta(\lambda_h^E h) = l_E(h)h^* + [a_E^-(h), a_E^+(h)] h_1^\perp.$$

Moreover, the set function  $[a_E^-, a_E^+]$  is upper semi-continuous in  $E$ .

3. For each  $c \in \mathcal{LF}_\beta(\lambda_h^E h)$ ,  $E \neq E_j$ , there is a unique  $c$ -minimal measure supported on  $\gamma_h^E$ .

4. For each  $c \in \mathcal{LF}_\beta(\lambda_h^{E_j} h)$ , there are two  $c$ -minimal measures supported on  $\gamma_h^{E_j}$  and  $c$ .

5. For  $E > 0$ , assume that the torus  $\mathbb{T}^2$  is not completely foliated by shortest closed  $g_E$ -geodesics in the homology  $h$ , then  $a_E^+(h) - a_E^-(h) > 0$  and the channel has non-zero width.

Assume that  $\gamma$  is a geodesic parametrized according to the Maupertuis principle. First, we note the following useful relation.

$$L(\gamma, \dot{\gamma}) + E = 2(E + U(\gamma)) = \sqrt{g_E(\gamma, \dot{\gamma})}, \quad (56)$$

where  $L$  denote the associated Lagrangian.

According to the theorem of Diaz Carneiro [28], the minimal measures for  $L$  is in one-to-one correspondence with the minimal measures of  $\frac{1}{2}g_E(\varphi, v)$ . On the other hand, any minimal measure  $\frac{1}{2}g_E$  with a rational rotation number is supported on closed geodesic. The following lemma characterizes minimal measures supported on a closed geodesic.

**Lemma C.1.** 1. Assume that  $c \in H^1(\mathbb{T}^2, \mathbb{R})$  is such that  $\alpha_H(c) = E > 0$ . Then for any  $h \in H_1(\mathbb{T}^2, \mathbb{Z})$ ,

$$l_E(h) - \langle c, h \rangle \geq 0.$$

2. Let  $\gamma$  be a closed geodesic of  $g_E$ ,  $E > 0$ , with  $[\gamma] = h \in H_1(\mathbb{T}^2, \mathbb{Z})$ . Let  $\mu$  be the invariant measure supported on the periodic orbit associated to  $\gamma$ . Then given  $c \in H^1(\mathbb{T}^2, \mathbb{R})$  with  $\alpha(c) = E$ ,

$$\mu \text{ is } c\text{-minimal if and only if } l_E(h) - \langle c, h \rangle = 0. \quad (57)$$

3. Let  $\gamma$  be a closed geodesic  $g_E$ ,  $E \geq 0$ , with  $[\gamma] = h \in H_1(\mathbb{T}^2, \mathbb{Z})$  and  $\alpha(c) = E$ . Then  $\gamma \subset \mathcal{A}_H(c)$  if and only if (57) holds.

*Proof.* Let  $\gamma$  be a closed geodesic of  $g_E$ ,  $E > 0$ , with  $[\gamma] = h$ . Assume that with the Maupertuis parametrization, the period of  $\gamma$  is  $T$ . Let  $\mu$  be the associated invariant measure, then  $\rho(\mu) = h/T$ . Assume that  $\alpha(c) = E$ , by definition, we have

$$\int L d\mu + E \geq \beta(h/T) + \alpha(c) \geq \langle c, h/T \rangle.$$

By (56), we have

$$\int L d\mu + E = \frac{1}{T} \int_0^T (L + E)(d\gamma) = \frac{1}{T} \int_0^T \sqrt{g_E(d\gamma)} = l_E(\gamma)/T.$$

Combine the two expressions, we have  $l_E(\gamma) - \langle c, h \rangle \geq 0$ . By choosing  $\gamma$  such that  $l_E(\gamma) = l_E(h)$ , statement 1 follows.

To prove statement 2, notice that if  $\mu$  is  $c$ -minimal, then  $\alpha(c) = E$  and the equality

$$\int L d\mu + E = \langle c, h/T \rangle$$

holds. Equality (57) follows from the same calculation as statement 1.

For  $E > 0$ ,  $\gamma \subset \mathcal{A}_H(c)$  if and only if  $\gamma$  is a minimal measure. Hence to prove statement suffices to prove for  $E = 0$ . In this case,  $\gamma$  can be parametrized as a homoclinic orbit.  $\gamma \subset \mathcal{A}(c)$  if and only if

$$\int_{-\infty}^{\infty} (L - c \cdot v + \alpha(c))(d\gamma) = 0.$$

Since

$$\int_{-\infty}^{\infty} (L - c \cdot v + \alpha(c))(d\gamma) = \int_{-\infty}^{\infty} (L + E)(d\gamma) - \langle c, h \rangle = l_E(h) - \langle c, h \rangle,$$

the statement follows.  $\square$

*Proof of Theorem 28.* By Lemma C.1, if there are two shortest geodesics  $\gamma_h^E$  and  $\bar{\gamma}_h^E$  for  $g_E$ , for any  $c$ , the invariant measure supported on  $\gamma_h^E$  is  $c$ -minimal if and only if the measure on  $\bar{\gamma}_h^E$  is  $c$ -minimal. This implies statement 1.

Statement 2 follows from the fact that  $\mathcal{LF}_\beta(\lambda_h^E h)$  is a closed convex set, and (57).  $\square$

Statement 3 and 4 follows directly from Lemma C.1.  $\square$

## C.2 Characterization of the channel and the Aubry sets

In this section we provide a precise characterization of the set

$$\mathcal{LF}_\beta(\lambda_h^E h) = l_E(h) h^* + [a_E^-(h), a_E^+(h)] h^\perp.$$

For each  $E > 0$ , we define

$$d_E^\pm(h) = \pm \inf_{n \rightarrow \infty} (l_E(nh \pm \bar{h}) - l_E(nh)),$$

where  $h, \bar{h}$  is a basis in  $H_1(T^2, \mathbb{R})$  and the dual of  $\bar{h}$  satisfies  $\langle \bar{h}^*, h \rangle = 0$  so we denote it  $h^\perp$ . Note that the sequence  $l_E(nh \pm \bar{h}) - l_E(nh)$  is decreasing, so the infimum coincides with the limit. We will omit dependence on  $h$  when it is not important.

**Proposition C.2.** *For each  $E > 0$ , we have*

$$d_E^\pm(h) = a_E^\pm(h).$$

*Proof.* We first show

$$d_E^-(h) \leq a_E^-(h) \leq a_E^+(h) \leq d_E^+(h).$$

Omit dependence on  $h$ . Denote  $c^+ = l_E(h)h^* + a_E^+h^\perp$ , by definition,  $l_E(h) - \langle c^+, h \rangle = 0$ . By Lemma C.1, statement 1, for  $n \in \mathbb{N}$ ,

$$\begin{aligned} 0 \leq l_E(nh + \bar{h}) - \langle c^+, nh + \bar{h} \rangle &= l_E(nh + \bar{h}) - nl_E(h) - \langle c^+, \bar{h} \rangle \\ &= l_E(nh + \bar{h}) - nl_E(h) - a_E^+. \end{aligned}$$

Take infimum in  $n$ , we have  $d_E^+ - a_E^+ \geq 0$ . Perform the same calculation with  $nh + \bar{h}$  replaced by  $nh - \bar{h}$ , we obtain  $0 \leq l_E(nh - \bar{h}) - nl_E(h) + a_E^-$ , hence  $a_E^- - d_E^- \geq 0$ .

We now prove the opposite direction. Take any  $c \in l_E(h)h^* + [d_E^-, d_E^+]h^\perp$ , we first show that  $\alpha(c) = E$ .

Take  $\rho \in \mathbb{Q}h + \mathbb{Q}\bar{h}$ , then any invariant measure  $\mu$  with rotation number  $\rho$  is supported on some  $[\gamma] = m_1h + m_2\bar{h}$  with  $m_1, m_2 \in \mathbb{Z}$ . Let  $T$  denote the period, by Lemma C.3 below,

$$\beta(\rho) + E = l_E(m_1h + m_2\bar{h})/T \geq \langle c, m_1h + m_2\bar{h} \rangle/T = \langle c, \rho \rangle.$$

Since  $\beta$  is continuous, we have  $\alpha(c) = \sup \langle c, \rho \rangle - \beta(\rho) \leq E$ , where the supremum is taken over all rational  $\rho$ 's. Since the equality is achieved at  $\rho = h$ , we conclude that  $\alpha(c) = E$ .

By Lemma C.1, statement 2, the measure supported on  $\gamma_h^E$  is  $c$ -minimal, and hence  $c \in \mathcal{LF}_\beta(\lambda_h^E h)$ .  $\square$

Recall  $h, \bar{h}$  form a basis in  $H_1(T^2, \mathbb{Z})$  and the dual of  $\bar{h}$  is perpendicular to  $h$  and denoted by  $h^\perp$ .

**Lemma C.3.** *For any  $c \in l_E(h)h^* + [d_E^-, d_E^+]h^\perp$  and  $m_1, m_2 \in \mathbb{Z}$ , we have*

$$l_E(m_1h + m_2\bar{h}) - \langle c, m_1h + m_2\bar{h} \rangle \geq 0.$$

*Moreover, if  $c \in l_E(h)h^* + (d_E^-, d_E^+)h^\perp$  and  $m_1, m_2 \neq 0$ , there exists  $a > 0$  such that*

$$l_E(m_1h + m_2\bar{h}) - \langle c, m_1h + m_2\bar{h} \rangle > a > 0.$$

*Proof.* The inequality for  $m_1 = 0$  or  $m_2 = 0$  follows from positive homogeneity of  $l_E$ . We now assume  $m_1, m_2 \neq 0$ .

If  $m_2 > 0$ , for a sufficiently large  $n \in \mathbb{N}$ , have

$$\begin{aligned} & l_E(m_1h + m_2\bar{h}) - \langle c, m_1h + m_2\bar{h} \rangle \\ &= l_E(m_1h + m_2\bar{h}) + l_E((nm_2 - m_1)h) - \langle c, nm_2h + m_2\bar{h} \rangle \\ &\geq l_E(m_2(nh + \bar{h})) - \langle c, m_2(nh + \bar{h}) \rangle = m_2(l_E(nh + \bar{h}) - \langle c, nh + \bar{h} \rangle) \\ &\geq l_E(nh + \bar{h}) - \langle c, nh + \bar{h} \rangle. \end{aligned}$$

Since

$$l_E(nh + \bar{h}) - \langle c, nh + \bar{h} \rangle = l_E(nh + \bar{h}) - nl_E(h) - \langle c, \bar{h} \rangle,$$

for  $c \in l_E(h)h^* + (d_E^-, d_E^+)h^\perp$ , then there exists  $a > 0$  such that for sufficiently large  $n$ ,

$$\lim_{n \rightarrow \infty} l_E(nh + \bar{h}) - nl_E(h) - \langle c, \bar{h} \rangle > a.$$

For  $m_2 < 0$ , we replace the term  $(nm_2 - m_1)h$  with  $(-nm_2 - m_1)h$  in the above calculation.  $\square$

We have the following characterization of the Aubry sets for the cohomologies contained in the channel.

**Proposition C.4.** *For any  $E > 0$  and  $c \in l_E(h)h^* + (d_E^-, d_E^+)h^\perp$ , we have*

$$\mathcal{A}_H(c) = \gamma_h^E$$

*if  $E$  is not a bifurcation value and*

$$\mathcal{A}_H(c) = \gamma_h^E \cup \bar{\gamma}_h^E$$

*if  $E$  is a bifurcation value.*

*Proof.* We first consider the case when  $E$  is not a bifurcation value. Since  $\gamma_h^E$  is the unique closed shortest geodesic, if  $\mathcal{A}_H(c) \supseteq \gamma_h^E$ , it must contain an infinite orbit  $\gamma^+$ . Moreover, as  $\gamma_h^E$  supports the unique minimal measure, the orbit  $\gamma^+$  must be biasumptotic to  $\gamma_h^E$ . As a consequence, there exists  $T_n, T'_n \rightarrow \infty$  such that  $\gamma^+(-T_n) - \gamma^+(T'_n) \rightarrow 0$ . By closing this orbit using a geodesic, we obtain a closed piece-wise geodesic curve  $\gamma_n$ . Moreover, since  $\gamma^+$  has no self-intersection, we can arrange it such that  $\gamma_n$  also have no self-intersection. We have

$$\int (L - c \cdot v + \alpha(c))(d\gamma_n) = \int (L + E)(d\gamma_n) - \langle c, [\gamma_n] \rangle = l_E(\gamma_n) - \langle c, [\gamma_n] \rangle.$$

By the definition of the Aubry set, and take limit as  $n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} l_E(\gamma_n) - \langle c, [\gamma_n] \rangle = 0.$$

Since  $\gamma_n$  has no self intersection, we have  $[\gamma_n]$  is irreducible. However, this contradicts w the strict inequality obtained in Lemma C.3.

We now consider the case when  $E$  is a bifurcation value, and there are two shortest geodesics  $\gamma_h^E, \bar{\gamma}_h^E$ . Assume by contradiction that  $\mathcal{A}_H(c) \supseteq \gamma_h^E \cup \bar{\gamma}_h^E$ . For mechanical systems on  $\mathbb{T}^2$ , the Aubry set satisfies an ordering property. As a consequence, there must exist two infinite orbits  $\gamma_1^+$  and  $\gamma_2^+$  contained in the Aubry set, where  $\gamma_1^+$  is forward asymptotic to  $\gamma_h^E$  and backward asymptotic to  $\bar{\gamma}_h^E$ , and  $\gamma_2^+$  is forward asymptotic to  $\bar{\gamma}_h^E$  and backward asymptotic to  $\gamma_h^E$ . Then there exists  $T_n, T'_n, S_n, S'_n \rightarrow \infty$  such that

$$\gamma_1^+(T'_n) - \gamma_2^+(-S_n), \gamma_2^+(S'_n) - \gamma_1^+(-T_n) \rightarrow 0$$

as  $n \rightarrow \infty$ . The curves  $\gamma_{1,2}^+, \gamma_h^E, \bar{\gamma}_h^E$  are all disjoint on  $\mathbb{T}^2$ . Similar to the previous case, we can construct a piecewise geodesic, non-self-intersecting closed curve  $\gamma_n$  with

$$\lim_{n \rightarrow \infty} \int (L - c \cdot v + \alpha(c))(d\gamma_n) = 0.$$

This, however, lead to a contradiction for the same reason as the first case.  $\square$

### C.3 The width of the channel

We show that under our assumptions, the “width” of the channel

$$d_E^+(h) - d_E^-(h) = \inf_{n \in \mathbb{N}} (l_E(nh + \bar{h}) - l_E(nh)) + \inf_{n \in \mathbb{N}} (l_E(nh - \bar{h}) - l_E(nh)),$$

is non-zero.

The following statement is a small modification of a theorem of Mather (see [59]), we provide a proof using our language.

**Proposition C.5.** *For  $E > 0$ , assume that the torus  $\mathbb{T}^2$  is not completely foliated by shortest closed  $g_E$ -geodesics in the homology  $h$ . Then*

$$d_E^+(h) - d_E^-(h) > 0.$$

**Remark C.1.** *This is the last item of Theorem 28.*

*Proof.* Let  $\mathcal{M}$  denote the union of all shortest closed  $g_E$ -geodesics in the homology  $h$ . We will show that  $\mathcal{M} \neq \mathbb{T}^2$  implies  $d_E^+(h) - d_E^-(h) > 0$ . Omit  $h$  dependence. For  $n \in \mathcal{N}$ , denote

$$d_n = (l_E(nh + \bar{h}) - l_E(nh)) + (l_E(nh - \bar{h}) - l_E(nh)).$$

Assume by contradiction that  $\inf d_n = \lim d_n > 0$ .

Let  $\gamma_0$  be a shortest geodesic in homology  $h$ . We denote  $\tilde{\gamma}_0$  its lift to the universal cover, and use “ $\leq$ ” to denote the order on  $\tilde{\gamma}$  defined by the flow. Let  $\gamma_1$  and  $\gamma_2$  be shortest curves in the homology  $nh + \bar{h}$  and  $nh - \bar{h}$  respectively, and let  $T_1$  and  $T_2$  be their periods.  $\gamma_i$  depends on  $n$  but we will not write it down explicitly.

Let  $\tilde{\gamma}_i$ ,  $i = 0, 1, 2$  denote a lift of  $\gamma_i$  to the universal cover. Using the standard curve shortening lemma in Riemannian geometry, it's easy to see that  $\tilde{\gamma}_i$  and  $\tilde{\gamma}_j$  may intersect at most once. Let  $a \in \gamma_0 \cap \gamma_1$  and lift it to the universal cover without changing its name. Let  $b \in \gamma_0 \cap \gamma_2$ , and we choose a lift in  $\tilde{\gamma}_0$  by the largest element such that  $b \leq a$ . We now choose the lifts  $\tilde{\gamma}_i$  of  $\gamma_i$ ,  $i = 1, 2$ , by the relations  $\tilde{\gamma}_1(0) = a$  and  $\tilde{\gamma}_2(T_2) = b$ .

We have for  $1 \leq k \leq 2n$ ,  $\tilde{\gamma}_2(T_2) + kh > \tilde{\gamma}_1(0)$  and

$$\tilde{\gamma}_2(0) + kh = b - (nh - \bar{h}) + kh \leq a + nh + \bar{h} = \tilde{\gamma}_1(T_1).$$

As a consequence,  $\tilde{\gamma}_2 + kh$  and  $\tilde{\gamma}_1$  has a unique intersection. Let

$$x_k = (\tilde{\gamma}_2 + kh) \cap \tilde{\gamma}_1, \quad \bar{x}_k = (\tilde{\gamma}_2 + (k-1)h) \cap (\tilde{\gamma}_1 - h).$$

We have  $x_k$  is in increasing order on  $\tilde{\gamma}_1$  and  $\bar{x}_k$  is in decreasing order on  $\tilde{\gamma}_2$ . Define

$$\tilde{\gamma}_k^* = \tilde{\gamma}_2|[\bar{x}_{k+1}, \bar{x}_k] * \tilde{\gamma}_1|[x_k, x_{k+1}],$$

and let  $\gamma_k^*$  be its projection to  $\mathbb{T}^2$ . We have  $[\gamma_k^*] = h$  and

$$\sum_{k=1}^{2n} l_E(\gamma_k^*) = l_E(\gamma_1) + l_E(\gamma_2).$$

Using  $l_E(\gamma_k) \geq l_E(h)$  and  $l_E(\gamma_1) + l_E(\gamma_2) \leq 2nl_E(h) + d_n$ , we obtain

$$l_E(h) \leq l_E(\gamma_k) \leq l_E(h) + d_n.$$

Any connected component in the complement of  $\mathcal{M}$  is diffeomorphic to an annulus. Pick one such annulus, and let  $b > 0$  denote the distance between its boundaries. Since  $\gamma_1$  intersects each boundary once, there exists a point  $y_n \in \gamma_1$  such that  $d(y_n, \mathcal{M}) = b/2$ . Since  $\gamma_1 \subset \bigcup_k \gamma_k^*$  there exists some  $\gamma_k^*$  containing  $y_n$ . By taking a subsequence if necessary, we may assume  $y_n \rightarrow y_*$ . Using the above discussion, we have

$$l_E(h) \leq \inf_{y_* \in \gamma, [\gamma]=h} l_E(\gamma) \leq \inf_n l_E(h) + d_n = l_E(h).$$

Using a similar argument as in the proof of Lemma A.11, we conclude that there exists a rectifiable curve  $\gamma_*$  containing  $y_*$  with  $l_E(\gamma_*) = l_E(h)$ , hence  $\gamma_*$  is a shortest curve. But  $y_* \notin \mathcal{M}$ , leading to a contradiction.  $\square$

Proposition C.5 clearly applies to the slow system as there are either one or two shortest geodesics.

## C.4 The case $E = 0$

We now extend the earlier discussions to the case  $E = 0$ . While the functions  $a_E^\pm$  is not defined at  $E = 0$ , the functions  $d_E^\pm$  is well defined at  $E = 0$ . Recall  $h, \bar{h}$  form a basis in  $H_1(T^2, \mathbb{Z})$  and the dual of  $\bar{h}$  is perpendicular to  $h$  and denoted by  $h^\perp$ .

**Proposition C.6.** *The properties of the channel and the Aubry sets depends on the type of homology  $h$ .*

1. *Assume  $h$  is simple and critical.*

- (a)  $d_0^+(h) - d_0^-(h) > 0$ .
- (b)  $l_0(h)h^* + [d_0^-(h), d_0^+(h)] h^\perp \subset \mathcal{LF}_\beta(0)$ .
- (c) For  $c \in l_0(h)h^* + [d_0^-(h), d_0^+(h)] h^\perp$ , we have  $\gamma_h^0 \subset \mathcal{A}_{H^s}(c)$ ;  
For  $c \in l_0(h)h^* + (d_0^-(h), d_0^+(h)) h^\perp$ , we have  $\gamma_h^0 = \mathcal{A}_{H^s}(c)$ .

2. *Assume  $h$  is simple and non-critical.*

- (a)  $d_0^+(h) - d_0^-(h) > 0$ .
- (b)  $l_0(h)h^* + [d_0^-(h), d_0^+(h)] h^\perp \subset \mathcal{LF}_\beta(0)$ .
- (c) For  $c \in l_0(h)h^* + [d_0^-(h), d_0^+(h)] h^\perp$ , we have  $\gamma_h^0 \cup \{0\} \subset \mathcal{A}_{H^s}(c)$ ;  
For  $c \in l_0(h)h^* + (d_0^-(h), d_0^+(h)) h^\perp$ , we have  $\gamma_h^0 \cup \{0\} = \mathcal{A}_{H^s}(c)$ .
- (d) The functions  $d_E^\pm(h)$  is right-continuous at  $E = 0$ .

3. *Assume  $h$  is non-simple and  $h = n_1 h_1 + n_2 h_2$ , with  $h_1, h_2$  simple.*

- (a)  $d_0^+(h) = d_0^-(h)$ . Moreover, let  $c^*(h) = l_E(h_1)h_1^* + l_E(h_2)h_2^*$ , where  $(h_1^*, h_2^*)$  is the dual basis to  $(h_1, h_2)$ , then

$$c^* = l_E(h)h^* + d_0^\pm(h)h^\perp,$$

where  $h^\perp$  is a unit vector perpendicular to  $h$ .

- (b)  $\gamma_{h_1}^0 \cup \gamma_{h_2}^0 = \mathcal{A}_{H^s}(c^*)$ .  
(c)  $d_0^+(h_1) - d_0^-(h_1) > 0$  with

$$l_E(h_1)h_1^* + d_0^+(h_1)h_2^* = c^*.$$

Before proving Proposition C.6, we first explain how the proof of Proposition C.5 can be adapted to work even for  $E = 0$ .

**Lemma C.7.** *Assume that there is a unique  $g_0$ -shortest geodesic in the homology  $h$ . Then*

$$d_0^+(h) - d_0^-(h) > 0.$$

*Proof.* We will try to adapt the proof of Proposition C.5. Let  $\gamma_0$ ,  $\gamma_1$  and  $\gamma_2$  be shortest geodesics in homologies  $h$ ,  $nh + \bar{h}$  and  $nh - \bar{h}$ , respectively. We choose an arbitrary parametrization for  $\gamma_i$  on  $[0, T]$ . Note that the parametrization is only continuous in general.

The proof of Proposition C.5 relies only on the property that lifted shortest geodesics intersects at most once. For  $E = 0$ , we will rely on a weaker property.

Let  $\tilde{\gamma}_i$  be the lifts to the universal cover  $\mathbb{R}^2$ . The degenerate point  $\{0\}$  lifts to the integer lattice  $\mathbb{Z}^2$ . Since  $g_0$  is a Riemannian metric away from the integers, using the shortening argument, we have: if  $\gamma_i$  intersect  $\gamma_j$  at more than one point, then either the intersections occur only at integer points, or the two curve coincide on a segment with integer end points.

Let  $a_0 \in \gamma_0 \cap \gamma_1$  and let  $\tilde{\gamma}_0$  and  $\gamma_1$  be lifts with  $\tilde{\gamma}_0(0) = \gamma_1(0) = a_0$ . If  $a_0 \notin \mathbb{Z}^2$ , then it is the only intersection between the two curves. If  $a_0 \in \mathbb{Z}^2$ , we define  $a'_0$  to be the largest intersection between  $\tilde{\gamma}_0|_{[0, T]}$  and  $\gamma_1$  according to the order on  $\tilde{\gamma}_0$ .  $a'_0$  is necessarily an integer point, and since  $a'_0 \in \tilde{\gamma}_0$ , there exists  $n_1 < n$  such that  $a'_0 - a_0 = n_0 h$ . Moreover, using the fact that  $\tilde{\gamma}_0$  is minimizing, we have

$$l_0(\tilde{\gamma}_0|[a_0, a'_0]) = l_0(\gamma_1|[a_0, a'_0]).$$

We now apply a similar argument to  $\tilde{\gamma}_0 + \bar{h}$  and  $\tilde{\gamma}_1$ . Let  $a_1 = \tilde{\gamma}_0(T) + \bar{h} = \tilde{\gamma}_1(T)$  and let  $a'_1$  be the smallest intersection between  $\tilde{\gamma}_0|_{(0, T]}$  and  $\tilde{\gamma}_1$ . Then there exists  $n_1 \in \mathcal{N}$ ,  $n_0 + n_1 < n$ , such that  $a_1 - a'_1 = n_1 h$ . Moreover,

$$l_0((\tilde{\gamma}_0 + \bar{h})|[a'_1, a_1]) = l_0(\tilde{\gamma}_1|[a'_1, a_1]).$$



Let  $\tilde{\eta}_1 = \tilde{\gamma}_1|_{[a'_0, a'_1]}$  and  $\eta_1$  be its projection. We have  $[\eta] = (n - n_0 - n_1)h + \bar{h} =: m_1h + \bar{h}$ , and

$$l_0(\eta_1) - m_1l_0(h) = l_0(\gamma_1) - nl_0(h).$$

The curve  $\eta_1$  has the property that it intersects  $\gamma_0$  only once. Apply the same argument to  $\gamma_2$ , we obtain a curve  $\eta_2$  with  $[\eta_2] = m_2h - \bar{h}$ , and

$$l_0(\eta_2) - m_2l_0(h) = l_0(\gamma_2) - nl_0(h).$$

To proceed as in the proof of Proposition C.5, we show that if  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$  are lifts of  $\eta_1$  and  $\eta_2$  with the property that

$$\tilde{\eta}_1(0), \tilde{\eta}_2(T) \in \{\tilde{\gamma}_0(t)\}, \quad \tilde{\eta}_1(T), \tilde{\eta}_2(0) \in \{\tilde{\gamma}_0(t) + \bar{h}\},$$

then  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$  intersects only once. Indeed, there are no integer points between  $\tilde{\gamma}_0$  and  $\tilde{\gamma}_0 + \bar{h}$ .

We have

$$l_0(\eta_1) - m_1l_0(h) + l_0(\eta_2) - m_2l_0(h) = d_n,$$

where  $d_n$  is as defined in Proposition C.5. Assume  $\inf d_n = 0$ , proceed as in the proof of Proposition C.5, we obtain curves  $[\gamma_k] = h$ , positive distance away from  $\gamma_0$ , such that

$$l_0(h) \leq l_0(\gamma_k) \leq l_0(h) + d_n.$$

This leads to a contradiction. □

*Proof of Proposition C.6. Case 1, h is simple and critical.*

(a) This follows from Lemma C.7.

(b) We note that Lemma C.3 depends only on positive homogeneity and sub-additivity of  $l_E(h)$ , and hence applies even when  $E = 0$ . We obtain for  $c \in l_0(h)h^* + [d_0^-(h), d_0^+(h)]\bar{h}^*$

$$l_0(h') - \langle c, h' \rangle \geq 0, \forall h' \in H_1(\mathbb{T}^2, \mathbb{Z}^2).$$

Since  $l_E(h)$  is strictly increasing, we obtain  $l_E(h') - \langle c, h' \rangle > 0$  for  $E > 0$ . By Lemma C.1, there are no  $c$ -minimal measures with energy  $E > 0$ . As a consequence,  $\alpha(c) = 0$ . Since  $\{0\}$  is a  $c$ -minimal measure with rotation number 0, we conclude  $l_0(h)h^* + [d_0^-(h), d_0^+(h)]\bar{h}^* \subset \mathcal{LF}_\beta(0)$ .

(c) Since we proved  $\alpha(c) = 0$ , the first conclusion follows from Lemma C.1. For the second conclusion, we verify that the proof of Proposition C.4 for non-bifurcation val applies to this case.

(d) The set function  $[d_E^-(h), d_E^+(h)]$  is upper semi-continuous at  $E = 0$  from the right, by definition. We will show that it is continuous. Assume by contradiction that

$$[\liminf_{E \rightarrow 0+} d_E^-(h), \limsup_{E \rightarrow 0+} d_E^+(h)] \subsetneq [d_0^-(h), d_0^+(h)].$$

Then there exists  $c \in l_0(h)h^* + (d_0^-(h), d_0^+(h))\bar{h}^*$  and

$$c(E) \notin l_E(h)h^* + [d_E^-(h), d_E^+(h)]h^\perp$$

such that  $c(E) \rightarrow c$ . By part (c), the Aubry set  $\mathcal{A}_{H^s}(c)$  supports a unique minimal measure. By Proposition 10.6, the Aubry set is upper semi-continuous in  $c$ . Hence any limit point of  $\mathcal{A}(c(E))$  as  $E \rightarrow 0$  is in  $\mathcal{A}(c)$ . This implies that  $\tilde{\mathcal{A}}(c(E))$  approaches  $\gamma_h^E$  as  $E \rightarrow 0$ . Since  $\gamma_h^E$  is the unique closed geodesic in a neighbourhood of itself (see Remark 3.3), we conclude that  $\tilde{\mathcal{A}}(c(E)) = \gamma_h^E$  for sufficiently small  $E$ . But this contradicts with  $c(E) \notin l_E(h)h^* + [d_E^-(h), d_E^+(h)]h^\perp$

*Case 2,  $h$  is simple and non-critical.*

(a) This follows from Lemma C.7.

(b) The proof is identical to case 1.

(c) For the first conclusion, we can directly verify that  $\gamma_h^0 \subset \mathcal{A}(c)$  and  $\{0\} \subset \mathcal{A}(c)$ .

For the second conclusion, we note that proof of Proposition C.4 for bifurcation values applies to this case.

*Case 3,  $h$  is non-simple with  $h = n_1h_1 + n_2h_2$ .*

(a) Assume that  $\bar{h} = m_1h_1 + m_2h_2$  for some  $m_1, m_2 \in \mathbb{Z}$ . For sufficiently large  $n \in \mathbb{N}$ , we have  $nh \pm \bar{h} \in \mathbb{N}h_1 + \mathbb{N}h_2$ . As a consequence,

$$\begin{aligned} l_0(nh \pm \bar{h}) - l_0(nh) &= (nn_1 \pm m_1)l_0(h_1) + (nn_2 \pm m_2)l_0(h_2) - (nn_1l_0(h_1) + nn_2l_0(h_2)) \\ &= \pm m_1l_0(h_1) \pm m_2l_0(h_2). \end{aligned}$$

We obtain  $d_0^+(h) - d_0^-(h) = 0$  by definition.

We check directly that

$$l_0(h) - \langle c^*, h \rangle = 0.$$

Since  $l_0(h)h^* + d_0^-(h)h^\perp = l_0(h)h^* + d_0^+(h)h^\perp$  is the unique  $c$  with this property. The second claim follows.

(b) We note that any connected component of the complement to  $\gamma_{h_1}^0 \cup \gamma_{h_2}^0$  is contractible. If  $\mathcal{A}_{H^s}(c)$  has other components, the only possibility is a contractible orbit bi-asymptotic to  $\{0\}$ . However, such an orbit can never be minimal, as the fixed point  $\{0\}$  has smaller action.

(c) The statement  $d_0^+(h_1) - d_0^-(h_1) > 0$  follows from part 1(a). for the second claim, we compute

$$d_0^+(h_1) = \inf_n l_0(nh_1 + h_2) - l_0(nh_2) = l_0(h_2)$$

and the claim follows.  $\square$

## D Transition between single and double resonance

Key Theorems 8 and 9 proved forcing equivalence for particularly chosen cohomology classes at single and double resonances. Given a single resonance  $\Gamma_k$ , the cohomology classes at single resonance is chosen to be a passage segment, namely, connected components of

$$\Gamma_k \setminus \bigcup_{p_0 \in \Sigma_K} U_{\bar{E}\sqrt{\epsilon}}(p_0),$$

where  $\Sigma_K$  is the collection of strong double resonances. Let  $p_0 \subset \Gamma_k \cap \Gamma_{k'}$  ( $k = (k_1, k_0)$ ,  $k' = (k'_1, k_0)$ ) be a strong double resonance, on the neighborhood  $U_{2\bar{E}\sqrt{\epsilon}}(p_0)$ , the cohomology class is chosen to be a curve

$$c_h(E) = p_0 + (B^T)^{-1} \bar{c}_h(E) \sqrt{\epsilon},$$

where

$$B = \begin{bmatrix} k_1 \\ k'_1 \end{bmatrix}, \quad \bar{c}_h(E) \in \text{int } \mathcal{LF}_\beta(\lambda_h^E h),$$

see sections B and C. The cohomology  $\bar{c}_h$  corresponds to the cohomology of the *slow mechanical system*  $H^s$ . In order to prove the forcing equivalence of all cohomology class, we will modify the single resonance cohomology class on the set

$$U_{2\bar{E}\sqrt{\epsilon}}(p_0) \setminus U_{\bar{E}\sqrt{\epsilon}}(p_0)$$

so that the choice of cohomology classes coincides with that of double resonance.

We first choose a particular parametrization for the curve  $\Gamma_k \cap U_{2\bar{E}\sqrt{\epsilon}}(p_0)$ .

**Lemma D.1.** *For sufficiently small  $\epsilon_0$ , there exists a function  $p_* : \mathbb{R} \times [0, \epsilon_0] \rightarrow \mathbb{R}^2$ , such that*

$$\Gamma_k \cap U_{2\bar{E}\sqrt{\epsilon}}(p_0) = p_0 + p_*(\lambda, \epsilon),$$

$p_*(0, \epsilon) = 0$ , and

$$p_*(\lambda, \epsilon) = (B \partial_{pp}^2 H_0(p_0))^{-1} \begin{bmatrix} 0 \\ \lambda \sqrt{\epsilon} \end{bmatrix} + O(\epsilon).$$

*Proof.*  $\Gamma_k$  is defined by the relation  $k \cdot (\partial_p H_0(p), 1) = 0$ . Using  $k \cdot (\partial_p H_0(p_0), 1) = 0$ , we have  $k_1 \cdot (\partial_p H_0(p) - \partial_p H_0(p_0)) = 0$ . We have

$$k_1 \cdot \partial_{pp}^2 H_0(p_0)(p - p_0) + O(p - p_0)^2 = 0, \text{ and}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} B \partial_{pp}^2 H_0(p_0)(p - p_0) + O(p - p_0)^2 = 0.$$

using  $p - p_0 = O(\sqrt{\epsilon})$ , we have

$$p - p_0 = (B \partial_{pp}^2 H_0(p_0))^{-1} \begin{bmatrix} 0 \\ \lambda \sqrt{\epsilon} \end{bmatrix} + O(\epsilon).$$

□

We now describe the modification of the cohomology class:

- On the set

$$\Gamma_k \cap U_{2\bar{E}\sqrt{\epsilon}}(p_0) \setminus U_{\bar{E}\sqrt{\epsilon}}(p_0),$$

we replace  $p_*(\lambda, \epsilon)$  by  $p_*(\lambda, 0)$ .

**Remark D.1.** *Both Key Theorem 4 and 5 applies with small perturbations to the cohomolgy class. For sufficiently small  $\epsilon_0 = \epsilon_0(\Gamma_*, H_0, \lambda)$  and  $0 < \epsilon \leq \epsilon_0$ , the theorems still apply for the modified cohomolgy.*

Denote  $I_*(p) = (B^T)^{-1}p_*(\lambda, 0)/\sqrt{\epsilon}$ , we have the following statement.

**Proposition D.2.** *There exists  $M > 0$  such that for  $\lambda > M$ , there exists  $\lambda' > 0$  such that*

$$I_*(\lambda) \in \text{int } \mathcal{LF}_\beta(\lambda' h).$$

**Remark D.2.** *We have the freedom to choose  $\bar{c}_h(E)$  as long as it is contained in the channel. Proposition D.2 implies that for sufficiently large  $E$ , we can choose  $c_h(E)$  to be on the curve  $I_*(\lambda)$ .*

*Proof.* The system  $H_\epsilon$  can be analyzed from two aspects, single resonance and double resonance. We first attempt to unify notations in both regimes.

When treated as single resonance, the system admits a single resonance normal form

$$N_\epsilon^{SR} = H_\epsilon \circ \Phi_\epsilon^{SR} = H_0 + \epsilon Z^{SR}(\theta^s, p) + O(\epsilon\delta),$$

where  $\theta^s = k \cdot (\theta, t)$  is the slow variable defined by the resonance  $\Gamma_k$ . The  $O(\cdot)$  term is in terms of the rescaled  $C^2$  norm  $C_I^2$ .

When treated as double resonance, we have two slow variables  $\theta^s$  and  $\theta^{sf} = k' \cdot (\theta, t)$ . The  $\theta^{sf}$  is considered fast in single resonance regime and is called  $\theta^f$  in Key Theorem 5. The system admits a double resonance normal form

$$N_\epsilon^{DR} = H_\epsilon \circ \Phi_\epsilon^{DR} = H_0 + \epsilon Z^{DR}(\theta^s, \theta^{sf}, p) + O(\epsilon^{\frac{3}{2}}).$$

Via a linear coordinate  $\Phi_L$  change and a rescaling  $\mathcal{S}$ , we have

$$N_\epsilon^{DR} \circ \Phi_L \circ \mathcal{S} = H_0(p_0)/\epsilon + H^s + O(\epsilon^{\frac{1}{2}}).$$

Working backwards, we have

$$N_\epsilon^{DR} - (H_0(p_0)/\epsilon + H^s) \circ \mathcal{S}^{-1} \circ \Phi_L^{-1} = O(\epsilon^{\frac{3}{2}}).$$

Define

$$\tilde{H}_\epsilon = (H_0(p_0)/\epsilon + H^s) \circ \mathcal{S}^{-1} \circ \Phi_L^{-1} \circ (\Phi_\epsilon^{DR})^{-1} \circ \Phi_\epsilon^{SR},$$

we have

$$\tilde{H}_\epsilon - N_\epsilon^{SR} = O(\epsilon^{\frac{3}{2}}).$$

We apply Key Theorem 5 to the system  $\tilde{H}_\epsilon$  with cohomology  $c = p_0 + p_*(\lambda, 0)$ , and obtain that the Mañe set  $\mathcal{N}(c)$  is a graph over the  $\theta^{sf}$  component. Taking it back via the coordinate change, we obtain that the Mañe set  $\mathcal{N}(\bar{c})$  for  $\bar{c} = I_*(\lambda)$  is a graph over the  $\theta^{sf}$  component as well. In this case, the only possibility is that it is a periodic orbit with homology  $h = (0, 1)$ . In view of Proposition C.5, this means  $\bar{c} \in \text{int } \mathcal{LF}_\beta(\lambda' h)$  for some  $\lambda' > 0$ .  $\square$

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## E Notations

Due to usage of a variety of techniques in this paper we feel it is useful to summarize notations of objects used in this paper.

$\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$	global angle variables.
$p = (p_1, p_2) \in B^2$	global action variables.
$\vec{k} = (\vec{k}_1, k_0)$	an integer vector defining a resonance, it belongs to $(\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$ .
$\Gamma = \Gamma_{\vec{k}}$	a resonance segment with resonant relation given by $\vec{k}$ .
$p_*^s(p^f)$	an implicit smooth function smoothly parametrizing $\Gamma = \{(p_*^s(p^f), p^f)\}$ , where $p^f$ varies in a certain interval.
$p_*(p^f) = p_*^s(p^f), p^f$	$H_0(p_*(p^f)) \equiv \text{const}$ , where again $p^f$ varies in a certain interval.
$\mathcal{S}^r$	the unit sphere of $C^r$ functions.
$\theta^s \in \mathbb{T}^2$	slow angle associated to a resonance $\Gamma_{\vec{k}}$ , $\theta^s = \vec{k} \cdot (\theta, t)$ .

$Z(\theta^s, p)$ $\Gamma^*$	a (single) averaged potential associated to a resonance $\Gamma_{\vec{k}}$ . a connected diffusion path consisting of subsegments of $\Gamma_j = \Gamma_{\vec{k}_j}$ 's.
$\mathcal{U}_{SR}$	the set of perturbations satisfying single resonant non-degeneracy conditions [G0]-[G2].
$\mathcal{U}_{SR}^\lambda$	the set of perturbations satisfying single resonant non-degeneracy conditions [G0]-[G2] with non-degeneracy parameter $\lambda > 0$ .
$\mathcal{U}_{DR}^E$	the set of perturbations satisfying double resonant non-degeneracy conditions [DR1]-[DR3], high energy.
$\mathcal{U}_{DR}^e$	the set of perturbations satisfying double resonant non-degeneracy conditions [A0]-[A4], low energy.
$c = c(H_0, r, \Gamma^*)$ $K = \left\lceil \frac{\delta^{4/(r-3)}}{c(H_0, r, \vec{k})} \right\rceil + 1$	additional constant for upper bounds. integer parameter dividing integer vectors $\vec{k}$ into those either producing or not strong double resonances and of non-degeneracy. See introduction.
$\Sigma_{K, \vec{k}}$ $(\theta^s, \theta^f) \in \mathbb{T}^s \simeq \mathbb{T}^2$	the set of punctures of $\Gamma_{\vec{k}}$ by strong double resonances. angle variable near a resonance segment, where $\theta^s = \vec{k}_1 \cdot \theta + k_0 t$ and $\theta^f = \vec{k}'_1 \cdot \theta + k'_0 t$ is transversal to it, i.e. $(\vec{k}_1, k_0) \nparallel (\vec{k}'_1, k'_0)$ .
$\lambda > 0$	quantitative non-degeneracy of global minima $\theta_i(p^f)$ of the averaged potential $Z(\theta_i^s, p_*^s(p^f), p^f)$ of the perturbation $H_1$ along the resonance $\Gamma_{\vec{k}}$ .
$\delta = \delta(H_0, H_1, r, \Gamma^*)$	parameter of extendability of maxima $\theta_i(p^f)$ beyond bifurcation values.
$b = b(H_0, H_1, r, \Gamma^*)$	some times we need an additional parameter $0 < b < \delta$ to characterize extendability.
$\kappa = \kappa(H^s) > 0$	quantitative non-degeneracy for conditions [A1]-[A4] to characterize persistence of normally hyperbolic invariant cylinders near double resonance for low energy (see section 3.4).
$[a_{min}, a_{max}]$	partition of resonant segment into bifurcation free intervals with a unique global minimum of averaged potential $Z(\theta^s, p)$ , i.e. $[a_{min}, a_{max}] = \cup_{j=1}^N [a_j, a_{j+1}]$ .
$\mathcal{C}_i^{(j)}$	a crumpled normally hyperbolic cylinder associated to a resonance $\Gamma_j$ and located “over” the $i$ -th interval of partition of $[a_{min}, a_{max}]$ .
$V_i^{(j)}$	a tube neighborhood of a crumpled normally hyperbolic cylinder $\mathcal{C}_i^{(j)}$ .

$A$	$2 \max_{p \in B^2, v \in \mathbb{R}^2,  v =1} \langle \partial_{pp}^2 H_0(p) v, v \rangle.$
$p_0 \in \Gamma_{\vec{k}} \cap \Gamma_{\vec{k}'}$	denotes a strong double resonance.
$D_\rho$	$D_\rho = \mathbb{T}^2 \times U_\rho(p_0) \times \mathbb{T} \times \mathbb{R}$ — the domain of validity of normal form at a double resonance, where $\rho = \bar{E} \sqrt{\varepsilon}$ .
$d$	denotes radius of the ball $B_d$ of the origin of the slow two torus $\mathbb{T}^s \times \mathbb{R}^2$ that decomposes dynamics into local and global.
$\varphi^s = (\varphi^{ss}, \varphi^{sf})$	angle coordinates near a strong double resonance defined for action $p$ being in a $O(\sqrt{\varepsilon})$ -neighborhood of $p_0$ .
$I^s = (I^{ss}, I^{sf})$	conjugate to angles near a strong double resonance defined for action $p$ being in a $O(\sqrt{\varepsilon})$ -neighborhood of $p_0$ .
$K(I^s) - U(\varphi^s)$	slow mechanical system obtained by averaging near $p_0$ and given by a kinetic energy $K(I^s)$ and potential $U(\varphi^s)$ , see (54).
$E_0 = E_0(H_0, H_1)$	small energy of this mechanical system with the saddle at the origin dominating behavior of homoclinics (see Key Theorem 3 item two).
$\bar{E}(H_0, H_1)$	large energy of this mechanical system with potential being small perturbation of kinetic energy.
$M = \bar{E}/2$	parameter of size of neighborhoods in section 7.1.
$0 < e < E_0$	very small energy of this mechanical system with hyperbolicity of the saddle at the origin dominating most contracting/expanding directions of NHIMs and certain non-degeneracies of the geodesic flow $\rho_0$ hold.
$h \in H_1(\mathbb{T}^s, \mathbb{Z})$	integer homology class of $\mathbb{T}^s = \mathbb{T}^2$ .
$g_E$	the Jacobi metric given by $g_E = \sqrt{2(E + U(\varphi^s))}$ .
$\gamma_h^E$	minimal geodesic of Mapertuis metric in homology class $h$ .
$E_0 > e$	small energy of the mechanical system so that dynamics of the origin start to dominate. In particular, there are no bifurcations of minimal geodesics $\gamma_h^E$ in $E$ .
$[E_0, \bar{E}]$	partition of slow energies $[E_0, \bar{E}] = \cup_{j=1}^N [E_j, E_{j+1}]$ into intervals with a unique globally minimizing geodesic (except the points).
$\mathcal{M}_h^{E_j, E_{j+1}}$	Normally hyperbolic invariant manifold with boundary for mechanical system, given by the union of minimal geodesics $\cup_{E \in [E_j - \delta, E_{j+1} + \delta]} \gamma_h^E$ .

$\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$	Normally hyperbolic weakly invariant manifold with boundary for the original Hamiltonian, where weakly invariant mean that the vector field of the original Hamiltonian is tangent to $\mathcal{M}_{h,\varepsilon}^{E_j, E_{j+1}}$ . This does not exclude possibility of “leak” through the boundary.
$\mathcal{M}_h^{E_0, s}$	Normally hyperbolic invariant manifold containing a critical simple loop for the mechanical system.
$\mathcal{M}_{h,\epsilon}^{E_0, s}$	Normally hyperbolic weakly invariant manifold containing a critical simple loop for the original system.
$o_\epsilon$	perturbed saddle periodic orbit near the double resonance.
$\mathcal{I}$	Involution for the mechanical system $(\varphi^s, I^s) \mapsto (\varphi^s, -I^s)$ .
$\alpha(c)$	Mather’s $\alpha$ -function. Sometimes subindex $H$ indicates dependence on the underlying Hamiltonian.
$\beta(h)$	Mather’s $\beta$ -function. Sometimes subindex $H$ indicates dependence on the underlying Hamiltonian.
$\mathcal{LF}_\beta$	the Legendre-Fenichel transform. It maps $H^1(M, \mathbb{R})$ into nonempty, compact, subsets of $H_1(M, \mathbb{R})$ .
$\mathbb{L}$	the Legendre diffeomorphism conjugating the Hamiltonian flow with the corresponding Euler-Lagrange flow.
$L_H$	the dual Lagrangian associated to a Tonelli Hamiltonian.
$\gamma_h^E$	a shortest geodesic in homology class $h$ and energy $E$ , where $g_E$ is the Jacobi metric associated to the mechanical system $H^s = K - U$ .
$T(\gamma_h^E)$	period of $\gamma_h^E$ under this parametrization.
$\lambda_h^E$	inverse of period $\lambda_h^E = 1/(T(\gamma_h^E))$ .
$a_E^\pm(h)$	functions characterizing width of the channel of cohomologies (see Theorem 28).
$d_E^\pm(h)$	width of the channel along $h^\perp$ direction, defined in (??). We also show $d_E^\pm(h) = a_E^\pm(h)$ .
$\bigcup_{E>0} \mathcal{LF}_\beta(\lambda_h^E h)$	the channel of cohomologies associated to an integer homology class $h \in H^1(\mathbb{T}^2, \mathbb{Z})$ . For types of channels see Figures 13 and 14).
$h, \bar{h}$	a basis of homology in $H_1(\mathbb{T}^2, \mathbb{Z})$ , with the dual basis $h^*, \bar{h}^*$ having property $\langle h, \bar{h}^* \rangle = 0$ .
$h^\perp$	denotes $\bar{h}^*$ to emphasise the latter condition.
$L_{H,c}$	the shifted Lagrangian $L = L_H - c \cdot v - \alpha_H(c)$ .
$A_{H,c}(x, t; y, s)$	the action functional minimizing action among curves connecting $(x, t), (y, s) \in M \times \mathbb{T}$ . See section 10.1 for definitions.
$\bar{c}_h(E)$	cohomology of the minimal geodesic for homology $h$ and energy $E$ of the mechanical system.



$c_h(E)$	cohomology of the original system $H_\varepsilon$ corresponding to $\bar{c}_h(E)$ .
$c_h^\lambda$	cohomology of the original system $H_\varepsilon$ corresponding to $\bar{c}_h(0)$ .
$c_h^*(h)$	a pinching point for a non-simple $h$ (see Figure 14).
$\bar{b}_{h_1}(E)$	modification of the cohomology path $\bar{c}_{h_1}(E)$ with simple homology class $h_1$ to satisfy certain conditions relative to non-simple homology $h$ (see Proposition 4.2).
$\bar{c}_{h_1}^\varepsilon(E) = \bar{b}_{h_1}(E)$	a different notation for modification of the cohomology path $\bar{c}_{h_1}(E)$ .
$\Phi_L$	linear rescaling near double resonance.
$\Gamma_i^{sr}$	choice of cohomology classes near a single resonance.
$\Gamma_i^{dr}$	choice of cohomology classes near a double resonance.
$\Gamma_{h,s}^{0,E_0/\bar{E}}$	choice of cohomology along a simple cylinder.
$\Gamma_{h,f}^{e,E_0/\bar{E}}$	choice of cohomology along a non-simple (flower) cylinder.
$\varphi_s^t(\theta, p)$	the time $(t - s)$ map of the Hamiltonian vector field $H$ with the initial time $s$ .
$\varphi^t(\theta, p)$	the time $(t - s)$ map of the Hamiltonian vector field $H$ with the initial time $0$ .
$c \in H^1(M, \mathbb{R})$	cohomology class.
$\widetilde{\mathcal{M}}(c) \subset TM$	the (discrete) Mather set with cohomology $c$ .
$\mathcal{M}(c) \subset M$	the (discrete) projected Mather set with cohomology $c$ .
$\widetilde{\mathcal{M}}_H(c) \subset TM \times \mathbb{T}$	the (continuous) Mather set with cohomology $c$ .
$\mathcal{M}_H(c) \subset M \times \mathbb{T}$	the (continuous) projected Mather set with cohomology $c$ .
$\widetilde{\mathcal{A}}(c) \subset TM$	the (discrete) Aubry set with cohomology $c$ .
$\mathcal{A}(c) \subset M$	the (discrete) projected Aubry set with cohomology $c$ .
$\widetilde{\mathcal{A}}_H(c) \subset TM \times \mathbb{T}$	the (continuous) Aubry set with cohomology $c$ .
$\mathcal{A}_H(c) \subset M \times \mathbb{T}$	the (continuous) projected Aubry set with cohomology $c$ .
$\widetilde{\mathcal{N}}(c) \subset TM$	the (discrete) Mañé set with cohomology $c$ .
$\mathcal{N}(c) \subset M$	the (discrete) projected Mañé set with cohomology $c$ .
$\widetilde{\mathcal{N}}_H(c) \subset TM \times \mathbb{T}$	the (continuous) Mañé set with cohomology $c$ .
$\mathcal{N}_H(c) \subset M \times \mathbb{T}$	the (continuous) projected Mañé set with cohomology $c$ .
$T_\eta : C^0(M, \mathbb{R}) \circlearrowleft$	the Lax-Oleinik mapping.
$\eta$	a closed one form on $T^*M$ , usually with cohomology class $c$ .
$u$	a semi-concave function.
$d\gamma(\tau)$	$d\gamma(\tau) = (\gamma(\tau), \dot{\gamma}(\tau), \tau)$ the one jet of a $C^1$ -curve $\gamma(\tau)$ .
$\mathcal{G}_{\eta,u}$	an overlapping pseudograph given by $\{(x, \eta_x + du_x) : x \in M \text{ such that } du_x \text{ exists}\}$ .

$c(\mathcal{G})$	cohomology class of a pseudograph $\mathcal{G} = \mathcal{G}_{\eta,u}$ for some closed one-form $\eta$ and a semi-concave function $u$ , given by cohomology of $\eta$ .
$\mathbb{S}$	the set of semi-concave functions on $M$ .
$\mathbb{P} = H^1(M, \mathbb{R}) \times \mathbb{S}/\mathbb{R}$	the set of overlapping pseudographs.
$\Phi : \mathbb{P} \longrightarrow \mathbb{P}$	a unique mapping in the space of pseudographs.
$h_c(x, y)$	a barrier function with a given cohomology $c \in H^1(M, \mathbb{R})$ , two points $x, y \in M$ and integer time increments. Sometimes subindex $H$ indicates dependence on the underlying Hamiltonian.
$h_c(x, t; y, s)$	a time-dependent barrier function with a given cohomology $c \in H^1(M, \mathbb{R})$ and two points $(x, t), (y, s) \in M \times \mathbb{T}$ . Sometimes subindex $H$ indicates dependence on the underlying Hamiltonian.
$O(c)$	a family of open sets $\{O(c)\}_c$ outside of proper cylinders $\mathcal{M}$ supporting perturbations making barrier functions $h_c$ generic (See section 11.4).
$[f]_{\omega_0}$	averaging of a function at a double resonance $\omega_0 = \omega_0(p_0) = \partial_p H_0(p_0)$ with $p_0 = \Gamma_{\vec{k}} \cap \Gamma_{\vec{k}'}$ .
$N_\epsilon := H_\epsilon \circ \Phi_\epsilon$	the perturbed Hamiltonian $H_\epsilon = H_0 + \epsilon H_1$ in the normal form.
$\Phi_L$	a linear symplectic change of coordinates at a double resonance $p_0$ .
$N_\epsilon^s = N_\epsilon \circ \Phi_L$	the Hamiltonian $N_\epsilon$ written in the slow angle coordinates $(\varphi^s, p^s)$ after the symplectic change of coordinates $\Phi_L$ .
$H_\epsilon^s = \mathcal{S}_2 \circ \mathcal{S}_1(N_\epsilon^s)$	the Hamiltonian $N_\epsilon^s$ after further rescalings $\mathcal{S}_2 \circ \mathcal{S}_1$ in actions and time. See section B.2 for details.
$h, h_1 \in H_1(\mathbb{T}^2, \mathbb{R})$	are homology classes. Usually $h$ is a non-simple homology and $h_1$ is simple (see definition 3.1).
$\mathbb{L}$	a uniform family of Tonelli Lagrangians.
$\mathbb{H}$	a uniform family of Tonelli Hamiltonians.

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